MASTER OF COMPUTER APPLICATION

MCA-25

DISCRETE MATHEMATICS AND OPTIMIZATION

Directorate of Distance Education

Guru Jambheshwar University of Science & Technology

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CONTENTS

SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
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Lesson No. 1	
SET THEORY	
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STRUCTURE

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

1. describe sets and their representations
2. identify empty set, finite and infinite sets
3. define subsets, super sets, power sets, universal set
4. describe the use of Venn diagram for geometrical description of sets
5. illustrate the set operations of union, intersection, difference and complement
6. know the different algebraic laws of set-operations
7. illustrate the application of sets in solving practical problems

1.2 INTRODUCTION

One of the widely used concepts in present day Mathematics is the concept of Sets. It is considered the language of modern Mathematics. The whole structure of Pure or Abstract Mathematics is based on the concept of sets. German mathematician Georg Cantor (1845-1918) developed the theory of sets and subsequently many branches of modern Mathematics have been developed based on this theory. In this unit, preliminary concepts of sets,set operations and some ideas on its practical utility will be introduced.

1.3 SETS AND THEIR REPRESENTATION

A set is a collection of *well-defined* objects. By well-defined, it is meant that given a particular collection of objects as a set and a particular object, it must be possible to determine whether that particular object is a member of the set or not.

The objects forming a set may be of any sort– they may or may not have any common property. Let us consider the following collections:

(i) the collection of the prime numbers less than 15 i.e.,2,3,5,7, 11,13

(ii) the collection of 0, a, SachinTendulkar ,the river Brahmaputra

(iii) the collection of the beautiful cities of India

(iv) the collection of great mathematicians.

Clearly the objects in the collections (i) and (ii) are well-defined. For example, 7 is a member of (i), but 20 is not a member of (i). Similarly, 'a' is a member of (ii), but M.S.Dhoni is not a member. So,the collections (i) and (ii) are sets. But the collections (iii) and (iv) are not sets, since the objects in these collections are not well-defined.

The objects forming a set are called elements or members of the set.Sets are usually denoted by capital letters A, B, C, ...; X, Y, Z, ..., etc., and the elements are denoted by small letters a,b,c,...;x,y,z,...,etc.

If 'a' is an element of a set A, then we write aεA which is read as' a belongs to the set A' or in short,' a belongs to A'. If 'a' is not an element of A, we write a ø A and we read as' a does not belong to A'. For example, let A be the set of prime number less than15. Then $2\in A, 3\in A, 5\in A, 7\in A, 11\in A, 1\notin A, 4\notin A, 17\notin A$, etc.

Representation of Sets: Sets are represented in the following two methods:

1. Roster or tabular method
2. Set-builder or Rule method

In the Roster method, the elements of a set are listed in any order, separated by commas and are enclosed within braces, For example, A = {2, 3, 5, 7, 11, 13}

B = {0, a Sachin Tendulkar, the river Brahmaputra} C = {1, 3, 5, 7, ...}

In the set C, the elements are all the odd natural numbers.We cannot list all the elements and hence the dots have been used showing that the list continues in definitely.

In the Rule method,a variable x is used to represent the elements of a set, where the elements satisfy a definite property, say P(x). Symbolically, the set is denoted by {x:P(x)} or {x|p(x)}.For example,

A = {x: x is an odd natural number} B = {x: x^2–3x+2 = 0}, etc.

If we write these two sets in the Roster method, we get,

A = {1, 3, 5, ...}

B = {1, 2}

Some Standard Symbols for Sets and Numbers: The following standard symbols are used to represent different sets of numbers:

N = {1, 2, 3, 4, 5, ...}, the set of natural numbers

Z= {..., –3, –2, –1, 0, 1, 2, 3, ...}, the set of integers

Q={x:x=$^p/_q$; p, qcZ,qs0},the set of rational numbers

R = {x: x is a real number},

the set of real numbers Z^+,Q^+,R^+respectively represent the sets of positive integers, positive rational numbers and positive real numbers. Similarly, Z^-, Q^-, R^- represent respectively the sets of negative integers, negative rational numbers and negative real numbers. Z^0, Q^0, R^0represent the sets of non-zero integers, non- zero rational numbers and non-zero real numbers.

1.3.1 THE EMPTYSET

Definition: A set which does not contain any element is called an empty set or a null set or a void set. It is denoted by ϕ.

The following sets are some examples of empty sets.

(i) the set {x: x^2 = 3 and x cQ}

(ii) the set of people in Assam who are older than 500 years

(iii) the set of real roots of the equation $x^2 + 4 = 0$

(iv) the set of Lady President of India born in Assam.

1.3.2 FINITE AND INFINITESETS

Let us consider the sets A = {1, 2, 3, 4, 5}and B = {1, 4, 7, 10, 13, ...}

If we count the members (all distinct) of these sets, then the counting process comes to an end for the elements of set A, where as for the elements of B, the counting process does not come to an end. In the first case we say that A is a finite set and in the second case, B is called an infinite set. A has finite number of elements and number of elements in B are infinite.

Definition: A set containing finite number of distinct elements so that the process of counting the elements comes to an end after a definite stage is called a finite set; otherwise, a set is called an infinite set.

Example: State which of the following sets are finite and which are infinite.

(i) the set of natural numbers N

(ii) the set of male persons of Assam as on January 1,2009.

(iii) the set of prime numbers less than20

(iv) the set of concentric circles in a plane

(v) the set of rivers on the earth.

Solution:

(i) N={1,2,3,...}is an infinite set

(ii) it is a finite set.

(iii) {2, 3, 5, 7, 11, 13, 17, 19} is a finite set

(iv) it is an infinite set

(v) it is a finite set.

1.3.3 EQUALSETS

Definition: Two sets A and B are said to be equal sets if every element of A is an element of B and every element of B is also an element of A. In other words, A is equal to B, denoted by A = B if A and B have exactly the same elements. If A and B are not equal, we write A ≠ B

Let us consider the sets A = {1, 2}

B = {x: (x–1) (x–2) = 0}

C = {x: (x–1) (x–2) (x–3) = 0}

Clearly B = {1, 2}, C = {1, 2, 3} and hence A = B, A sC, B sC.

Example:Find the equal and unequal sets:

(i) A = {1, 4,9}

(ii) B = $\{1^2, 2^2, 3^3\}$

(iii) C={x:x is a letter of the word TEAM}

(iv) D={x:x is a letter of the word MEAT}

(v) E = {1, {4},9}

Solution: A = B, C = D, A ≠ C, A ≠ D, A ≠ E, B ≠ C, B ≠ D, B ≠ E, C ≠ E, D ≠ E

1.3.4 SUBSETS, SUPERSETS, PROPERSUBSETS

Let us consider the sets A = {1, 2, 3}, B = {1, 2, 3, 4} and C = {3, 2, 1}. Clearly, every element of A is an element of B, but A is not equal toB.Again, every element of A is an element of C, and also A is equal to C. In both cases, we say that A is a subset of B and C.In particular, we say that A is a proper subset of B, but A is not a proper subset of C.

Definition: If every element of a set A is also an element of another set B, then A is called a subset of B, or A is said to be contained in B, and is denoted by A ⊆B. Equivalently, we say that B contains A or B is a super set of A and is denoted by B ⊇A.

Symbolically, $A \subseteq B$ means that for all x, if $x \in A$ then $x \in B$.

If A is a subset of B, but the re exists at least one element in B which is not in A, then A is called a proper subset of B, denoted by $A \subset B$.In other words, $A \subset B \Leftrightarrow (A \subseteq B$ and $A \neq B)$.

The symbol '$\Leftrightarrow$' stands for 'logically implies and is implied by'. Some examples of proper subsets are as follows: $N \subset Z$, $N \subset Q$, $N \subset R$, $Z \subset Q$, $Z \subset R$, $Q \subset R$.

It should be noted that any set A is a subset of itself, that is, $A \subseteq A$. Also, the null set ϕ is a subset of every set, that is, $\phi \subseteq A$ for any set A. Because, if $\phi \subseteq A$, then there must exist an element $x \in \phi$ such that $x \notin A$. But $x \notin \phi$, hence we must accept that $\phi \subseteq A$. Combining the definitions of equality of sets and that of subsets.

we get $A = B \Leftrightarrow (A \subseteq B$ and $B \subseteq A)$

1.3.5 POWERSET

Let us consider a set A = {a, b}. A question automatically comes to our mind– 'What are the subsets of A?' The subsets of A are ϕ, {a}, {b} and A itself. These subsets, taken as elements, again form a set. Such a set is called the power set of the given set A.

Definition: The set consisting of all the subsets of a given set A as its elements, is called the power set of A and is denoted by P(A) or 2^A. Thus, P(A) or $2^A = \{X: X \subseteq A\}$

Clearly,

(i) $P(\phi) = \{\phi\}$

(ii) if A = {1}, then P^A= {\$, {1}}

(iii) if A = {1, 2}, then P^A= {\$, {1}, {2}, A}

(iv) if A = {1, 2, 3}, then P^A= {\$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, A}

From these examples we can conclude that if a set A has n elements,then P(A) has 2^n elements.

1.3.6 UNIVERSALSET

A set is called a Universal Set or the Universal discourse if it contains all the sets under consideration in a particular discussion. A universal set is denoted by U.

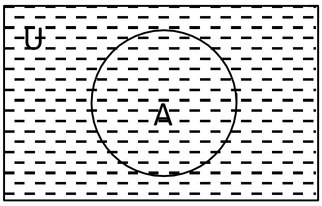
Example:

- For the sets {1, 2, 3}, {3, 7, 8}, {4, 5, 6, 9}We can take U = {1, 2, 3, 4, 5, 6, 7, 8, 9}
- In connection with the sets N, Z, Q we can take R as the universal set.
- In connection with the population in India, the set of all people in India is the universal set, etc.

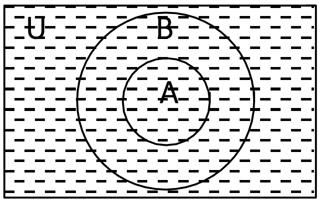
1.4VENNDIAGRAM

Simple plane geometrical areas are used to represent relationships between sets in meaningful and illustrative ways. These diagrams are called Venn-Euler diagrams, or simply theVenn-diagrams.

In Venn diagrams, the universal set U is generally represented by a set of points in a rectangular area and the subsets are represented by circular regions within the rectangle, or by any closed curve within the rectangle. As an illustration Venn diagrams of $A \subset U$,$A \subset B \subset U$ are given below:



$A \subset U$



$A \subset B \subset U$

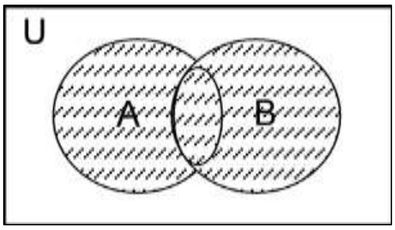
Similar Venn diagrams will be used in subsequent discussions illustrating different algebraic operations on sets.

1.5 SETOPERATIONS

We know that given a pair of numbers x and y, we can get new numbers x+y, x–y, xy, $^x/_y$ (with $y \neq 0$) under the operations of addition, subtraction, multiplication and division. Similarly, given the two sets A and B we can form new sets under set operations of union, intersection, difference and complements. We will now define these set operations, and the new sets thus obtained will be shown with the help of Venn diagrams.

1.5.1 UNION OF SETS

Definition: The union of two sets A and B is the set of all elements which are members of set A or set B or both. It is denoted by $A \cup B$, read as 'A union B' where '$\cup$' is the symbol for the operation of 'union'. Symbolically we can describe $A \cup B$ as follows:



$A \cup B = \{x : x \in A \text{ or } x \in B\}$

A ∪B (Shaded)

It is obvious that $A \subseteq A \cup B$, $B \subseteq A \cup B$

Example 1: Let A = {1, 2, 3, 4}, B = {2, 4, 5, 6}

Then $A \cup B$ = {1, 2, 3, 4, 5, 6}

Example2: Let Q be the set of all rational numbers and K be the set of all irrational numbers and R be the set of all real numbers. Then $Q \cup K = R$

Identities: If A, B, C be any three sets, then

(i) $A \cup B = B \cup A$

(ii) $A \cup A = A$

(iii) A∪$= A

(iv) $A \cup U = U$

(v) $(A \cup B) \cup C = A \cup (B \cup C)$

Proof:

(i) $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\}$

$= B \cup A$

(ii) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(iii) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

(iv) $A \cup U = \{x : x \in A \text{ or } x \in U\}$

$= \{x : x \in U\}$, since $A \subset U$

$= U$

(v) $(A \cup B) \cup C = \{x : x \in A \cup B \text{ or } x \in C\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$

$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C\}$

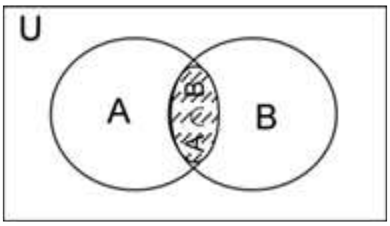
$= \{x : x \in A \text{ or } x \in B \cup C\}$

$= A \cup (B \cup C)$

1.5.2 INTERSECTION OF SETS

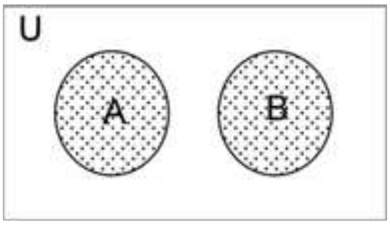
Definition: The intersection of two sets A and B is the set of all elements which are members of both A and B. It is denoted by $A \cap B$, read as 'A intersections B', where '$\cap$' is the symbol for the operation of 'intersection'. Symbolically we can describe it as follows:

$A \cap B = \{x : x \in A \text{ and } x \in B\}$



A∩B (Shaded)

From definition it is clear that if A and B have no common element, then $A \cap B = \phi$. In this case, the two sets A and B are called disjoint sets.



$A \cap B = \phi$

It is obvious that $A \cap B \subseteq A, A \cap B \subseteq B$.

Example1: LetA={a,b,c,d},B={b,d,4,5} Then A∩B={b,d}

Example 2: Let A = {1, 2, 3}, B = {4, 5, 6}Then $A \cap B = \phi$.

Identities:

(i) $A \cap B = B \cap A$

(ii) $A \cap A = A$

(iii) $A \cap \phi = \phi$

(iv) $A \cap U = A$

(v) $(A \cap B) \cap C = A \cap (B \cap C)$

(vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

(i) $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= B \cap A$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\}$

$= \{x : x \in A\}$

$= A$

(iii) Since ϕ has no element, so A and ϕ have no common element.

Hence $A\cap\phi=\phi$

(iv) $A\cap U \quad =\{x : x\in A \text{ and } x\in U\}$

$= \{x : x \in A\}$, since $A\subset U$

$= A$

(v) $(A\cap B)\cap C=\{x : x\in A\cap B \text{ and } x\in C\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } x \in B \cap C\}$

$= A\cap(B \cap C)$

(vi) $x\in A\cap(B\cup C) \Leftrightarrow \in A \text{ and } x\in(B\cup C)$

$\Leftrightarrow \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x\in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Leftrightarrow \in(A\cap B) \text{ or } x \in(A \cap C)$

$\Leftrightarrow \in(A \cap B) \cup(A \cap C)$

So, $A \cap(B\cup C) \in(A\cap B) \cup(A \cap C)$

and $(A \cap B) \cup(A \cap C) \in A \cap(B \cup C)$.

Hence, $A \cap(B \cup C) = (A\cap B) \cup(A\cap C)$.

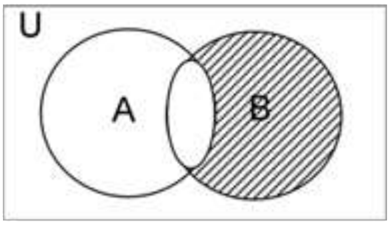
Similarly, it can be proved that $A\cup(B \cap C) = (A \cup B) \cap(A \cup C)$.

1.5.3 DIFFERENCE OF SETS

Definition: The difference of two sets A and B is the set of all elements which are members of A, but not of B. It is denoted by A–B.

Symbolically, $A - B = \{x: x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x: x \in B \text{ and } x \notin A\}$



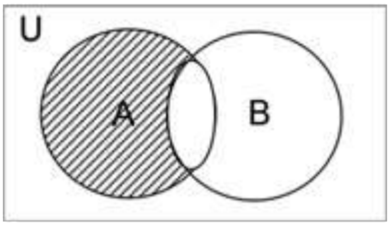


FIG 1.1A –B(SHADED) **FIG 1.2B – A (SHADED)**

Example: Let A = {1, 2, 3, 4, 5}, B = {1, 4, 5}, C = {6, 7, 8}

Then A – B = {2, 3}

A – C = A

B – C = B

B – A = ϕ

Properties:

(i) $A - A = \phi$

(ii) $A - B \subseteq A, B - A \subseteq B$

(iii) $A-B, A\cap B, B-A$ are mutually disjoint and $(A-B)\cup(A\cap B)\cup(B-A)=A\cup B$

(iv) $A-(B\cup C)=(A-B)\cap(A-C)$

(v) $A-(B\cap C)=(A-B)\cup(A-C)$

Proof:We prove (iv),others are left as exercises.

$x \in A-(B \cup C) \Leftrightarrow \in A$ and $x \notin (B \cup C)$

$\Leftrightarrow \in A$ and ($x \notin B$ and $x \notin C$)

$\Leftrightarrow$($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

$\Leftrightarrow \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow \in (A - B) \cap (A - C)$

So, $A-(B \cup C) \subseteq (A-B) \cap (A-C), (A-B) \cap (A-C) \subseteq A-(B \cup C)$

Hence, $A-(B \cup C)=(A-B) \cap (A-C)$.

1.5.4 COMPLEMENT OF A SET

Definition:If U be the universal set of a set A, then the set of all those elements in U which are not members of A is called the Compliment of A, denoted by A^c or A'.

Symbolically, $A'=\{x : x \in U \text{ and } x \notin A\}$.

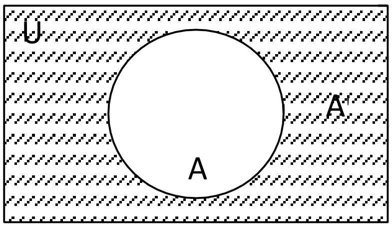


Fig 1.3 A′ (Shaded)

Clearly, $A'=U-A$.

Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {2, 4, 6, 8}

Then $A'=\{1,3,5,7,9\}$

Identities:

(i) $U'=\phi, \phi'=U$

(ii) $(A')'=A$

(iii) $A \cup A'=U, A \cap A'=\phi$

(iv) $A-B=A\cap B', B-A=B\cap A'$

(v) $(A\cup B)'=A'\cap B', (A\cap B)'=A'\cup B'$

Proof: We prove $(A\cup B)'=A'\cap B'$. The rest are left as exercises.

$(A\cup B)'=\{x : x\in U \text{ and } x\notin A\cup B\}$

$= \{x : x \in U \text{ and } (x\notin A \text{ and } x \notin B\}$

$= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\}$

$=\{x : x\in A' \text{ and } x\in B'\}$

$=A'\cap B'$

1.6LAWS OF THE ALGEBRA OFSETS

In the preceding discussions we have stated and proved various identities under the operations of union, intersection and complement of sets. These identities are considered as Laws of Algebra of Sets. These laws can be directly used to prove different propositions on Set Theory. These laws are given below:

1. **Idempotent laws:** $A\cup A=A, A\cap A=A$
2. **Commutative laws:** $A\cup B=B\cup A, A\cap B=B\cap A$
3. **Associative laws:** $A\cup(B\cup C)=(A\cup B)\cup C$,

   $A \cap(B \cap C) = (A \cap B) \cap C$
4. **Distributive laws:** $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$,

   $A \cap(B \cup C) = (A \cap B) \cup(A \cap C)$
5. **Identity laws:** $A\cup\$=A, A\cup U=U$

   $A \cap U = A$, $A \cap\phi= \phi$
6. **Complement laws:** $A\cup A'=U$, $AnA'=\phi$

   $(A')'=A, U'=\phi, \phi'=U$

7. DeMorgan's laws: $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$.

Let us illustrate the application of the laws in the following examples:

Example 1: Prove that $A \cap (A \cup B) = A$

Solution: $A \cap (A \cup B) = (A \cup \phi) \cap (A \cup B)$, using identity law

$= A \cup (\phi \cap B)$, using distributive law

$= A \cup (B \cap \phi)$, using commutative law

$= A \cup \phi$, using identity law

$= A$, again using identity law

Example2: Prove that $A \cap (A' \cup B) = A \cap B$

Solution: $A \cap (A' \cup B) = (A \cap A') \cup (A \cap B)$, using distributive law

$= \phi \cup (A \cap B)$, using complement law

$= (A \cap B) \cup \phi$, using commutative law

$= A \cap B$, using identity law

1.7 COUNTING PRINCIPLES

A set **S** is countable if **S** is finite or if the elements of **S** can be arranged as a sequence, in which case **S** is said to be countably infinite; otherwise **S** is said to be uncountable. The set E of even integers is countably infinite, whereas one can prove that the unit interval **I** = [0, 1] is uncountable.

Counting Elements in Finite Sets

The notation n(S) or |S| will denote the number of elements in a set S. Thus n(A) = 26, where A is the letters in the English alphabet, and n(D) = 7, where D is the days of the week. Also n(Φ) = 0 since the empty set has no elements.

The following theorem applies.

Theorem: Suppose A and B are finite disjoint sets. Then A ∪ B is finite and

$$n(A \cup B) = n(A) + n(B)$$

This theorem may be restated as follows:

Suppose S is the disjoint union of finite sets A and B. Then S is finite and

$$n(S) = n(A) + n(B)$$

Proof: In counting the elements of A ∪ B, first count those that are in A. There are n(A) of these. The only other elements of A ∪ B are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are n(B) elements that are in B but not in A. Therefore, n(A ∪ B) = n(A) + n(B).

For any sets A and B, the set A is the disjoint union of A\B and A ∩ B. Thus theorem gives us the following useful result.

Result 1: Let A and B be finite sets. Then

$n(A \setminus B) = n(A) - n(A \cap B)$

For example, suppose an art class A has 25 students and 10 of them are taking a biology class B. Then the number of students in class A which are not in class B is:

$n(A \setminus B) = n(A) - n(A \cap B) = 25 - 10 = 15$

Given any set A, recall that the universal set **U** is the disjoint union of A and AC. Accordingly, Theorem also gives the following result.

Result 2: Let A be a subset of a finite universal set **U**. Then

n(AC) = n(**U**) − n(A)

For example, suppose a class **U** with 30 students has 18 full-time students. Then there are 30−18 = 12 part-time students in the class **U**.

1.7.1 Inclusion–Exclusion Principle

There is a formula for $n(A \cup B)$ even when they are not disjoint, called the Inclusion–Exclusion Principle. Namely:

Theorem (Inclusion–Exclusion Principle) : Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting $n(A \cap B)$ (exclusion). since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

Result 3: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

EXAMPLE : Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:

(a) only on list A, (b) only on list B,

(c) on list A or B (or both), (d) on exactly one list.

(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$.

By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

(d) By (a) and (b), 10 + 15 = 25 names are only on one list; that is, $n(A\ ^{\theta}\ B) = 25$.

1.8 CLASSES OF SETS

CLASSES OF SETS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets.Whenever such a situation occurs, to avoid confusion, we will speak of a class of sets or collection of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of subclass or subcollection.

EXAMPLE Suppose S = {1, 2, 3, 4}.

(a) Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

That is, the elements of A are the sets {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, and {2, 3, 4}.

(b) Let B be the class of subsets of S, each which contains 2 and two other elements of S. Then

$$B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of B are the sets {1, 2, 3}, {1, 2, 4}, and {2, 3, 4}. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

1.9 PARTITIONS OF SET

Partitions

Let S be a nonempty set. A partition of S is a subdivision of S into non overlapping, nonempty subsets.

Precisely, a partition of S is a collection {Ai } of nonempty subsets of S such that:

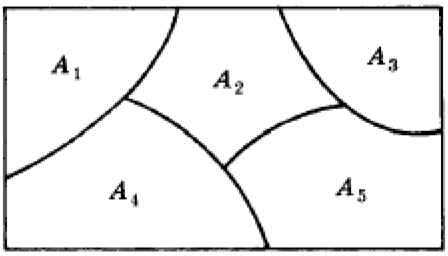
(i) Each a in S belongs to one of the Ai .

(ii) The sets of {Ai } are mutually disjoint; that is, if

$$Aj \neq Ak \text{ then } Aj \cap Ak = \Phi$$

The subsets in a partition are called cells. Figure given below is a Venn diagram of a partition of the rectangular set

S of points into five cells, A1, A2,A3,A4,A5.



EXAMPLE: Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) [{1, 3, 5}, {2, 6}, {4, 8, 9}]

(ii) [{1, 3, 5}, {2, 4, 6, 8}, {5, 7, 9}]

(iii) [{1, 3, 5}, {2, 4, 6, 8}, {7, 9}]

Then (i) is not a partition of *S* since 7 in *S* does not belong to any of the subsets. Furthermore, (ii) is not a partition of *S* since {1, 3, 5} and {5, 7, 9} are not disjoint.

On the other hand, (iii) is a partition of *S*.

1.10 MULTI SET

What is a Multiset?

A **multiset** in mathematics is a generalization of the concept of a set. It's a collection of unordered numbers (or other elements), where every element x occurs a finite number of times.

The difference between sets and multisets is in how they address multiples: a set includes any number at most once, while a multiset allows for multiple instances of the same number. There is just one set with elements a and b, the set {a,b}, but there are many multisets: {a, b, b}, {a, a, b}, and {a, a, a, a, b, b} are just a few.

Multiplicity:

The **multiplicity** of an element x in a multiset is just the number of times that element appears in the set. For instance, in the multiset {3, 3, 4, 5, 6} the element 3 has multiplicity 2. The elements 4, 5, and 6 all have multiplicity 1.

If we know the elements included and the multiplicity for each of them we know everything about a multiset. A multiset which includes just 4 with multiplicity 2, 5 with multiplicity 7 and 99 with multiplicity 2 can be written as {4, 4, 5, 5, 5, 5, 5, 99, 99}. Order doesn't matter, so this is the same as {4, 99, 5, 4, 99, 5, 5, 5, 5 }

To distinguish between sets and multisets, a notation that incorporates square brackets is sometimes used: the multiset {a, a, b} can be denoted as [a, a, b].

The cardinality of a multiset is constructed by summing up the multiplicities of all its elements. For example, in the multiset {a, a, b, b, b, c} the multiplicities of the members a, b, and c are respectively 2, 3, and 1, and therefore the cardinality of this multiset is 6.

Operations on Multisets

1. Union of Multisets: The Union of two multisets A and B is a multiset such that the multiplicity of an element is equal to the maximum of the multiplicity of an element in A and B and is denoted by A ∪ B.

Example:

Let A = {l, l, m, m, n, n, n, n}

B = {l, m, m, m, n},

A ∪ B = {l, l, m, m, m, n, n, n, n}

2. Intersections of Multisets: The intersection of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the minimum of the multiplicity of an element in A and B and is denoted by A ∩ B.

Example:

Let A = {l, l, m, n, p, q, q, r}

B = {l, m, m, p, q, r, r, r, r}

$A \cap B = \{l, m, p, q, r\}$.

3. Difference of Multisets: The difference of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is +ve, and is equal to 0 if the difference is 0 or negative

Example:

Let $A = \{l, m, m, m, n, n, n, p, p, p\}$

$B = \{l, m, m, m, n, r, r, r\}$

$A - B = \{n, n, p, p, p\}$

4. Sum of Multisets: The sum of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the sum of the multiplicity of an element in A and B

Example:

Let $A = \{l, m, n, p, r\}$

$B = \{l, l, m, n, n, n, p, r, r\}$

$A + B = \{l, l, l, m, m, n, n, n, n, p, p, r, r, r\}$

5. Cardinality of Sets: The cardinality of a multiset is the number of distinct elements in a multiset without considering the multiplicity of an element

Example:

$A = \{l, l, m, m, n, n, n, p, p, p, p, q, q, q\}$

The cardinality of the multiset A is 5.

1.11 CHECK YOUR PROGRESS

1. Express the following sets in Roster method:

   (i) A={x: x is a day of the week}

   (ii) B = {x: x is a month of the year}

(iii) C = {x: x^3–1 =0}

(iv) D = {x: x is a positive divisor of100}

(v) E={x:x is a letter of the word ALGEBRA}

2. Which of the following sets are equal?

A = {x | x2 − 4x + 3 = 0}, C= {x | x ∈ **N**, x <3}, E= {1, 2}, G= {3, 1},

B = {x | x2 − 3x + 2 = 0}, D= {x | x ∈ **N**, x is odd, x <5}, F = {1, 2, 1}, H= {1, 1, 3}.

3. List the elements of the following sets if the universal set is **U** = {*a*, *b*, *c*, …, *y*, *z*}.

Furthermore, identify which of the sets, if any, are equal.

A = {x | x is a vowel}, C= {x | x precedes f in the alphabet},

B = {x | x is a letter in the word "little"}, D= {x | x is a letter in the word "title"}.

4. Let *A* = {1, 2, …, 8, 9}, *B* = {2, 4, 6, 8}, *C* = {1, 3, 5, 7, 9}, *D* = {3, 4, 5}, *E* = {3, 5}.

Which of the these sets can equal a set *X* under each of the following conditions?

(a) *X* and *B* are disjoint. (c) *X* ⊆*A* but *X* _⊂*C.*

(b) *X* ⊆*D* but *X* _⊂*B.* (d) *X* ⊆*C* but *X* _⊂*A.*

5. Write true or false:

(i) 5⊂N (ii) ½ ⊂ Z (iii) –1 ⊂Q

(iv) 2 ⊂ R (v) -1 ⊂R (vi) –3 ⊄N

6. Consider the universal set **U** = {1, 2, 3, …, 8, 9} and sets *A* = {1, 2, 5, 6}, *B* = {2, 5, 7}, C= {1, 3, 5, 7, 9}. Find:

(a) *A* ∩ *B* and *A* ∩ *C* (c) *A*C and CC (e) *A* Φ *B* and *A* Φ *C*

(b) *A* ∪ *B* and *B* ∪ *C* (d) *A*\B and *A*\C (f) *(A* ∪ *C)**B* and *(B* Φ*C)**A*

7. Find the empty sets, finite and infinite sets:

(i) the set of numbers divisible by zero

(ii) the set of positive integers less than 15 and divisible by 17

(iii) the set of planets of the solar system

(iv) the set of positive integers divisible by 4

(v) the set of co-planer triangles

(vi) the set of Olympians from Assam participating in 2008,Beijing Olympic.

8. Examine the equality of the following sets:

(i) A = {2, 3}, B = {x: x^2–5x+6 =0}

(ii) A = {x: x is a letter of the word WOLF} B = {x: x is a letter of the word FLOW}

(iii) A = {a, b, c}, B = {a, {b,c}}

9. If A={3n:n∈N}, B={n:n∈N,n<20} then find A∩B, B–A.

10. Find the power set *P(A)* of *A* = {1, 2, 3, 4, 5}.

11. In a survey of 120 people, it was found that:

65 read Newsweek magazine, 20 read both Newsweek and Time,

45 read Time, 25 read both Newsweek and Fortune,

42 read Fortune, 15 read both Time and Fortune,

8 read all three magazines.

(*a*) Find the number of people who read at least one of the three magazines.

(*b*) Find the number of people who read exactly one magazine.

12. Let *A* = [{1, 2, 3}, {4, 5}, {6, 7, 8}]. (*a*) List the elements of *A*; (*b*) Find *n(A)*.

13. Let *S* = {*a*, *b*, *c*, *d*, *e*, *f* , *g*}. Determine which of the following are partitions of *S*:

(*a*) *P*1 = [{*a*, *c*, *e*}, {*b*}, {*d*, *g*}], (*c*) *P*3 = [{*a*, *b*, *e*, *g*}, {*c*}, {*d*, *f* }],

(*b*) *P*2 = [{*a*, *e*, *g*}, {*c*, *d*}, {*b*, *e*, *f* }], (*d*) *P*4 = [{*a*, *b*, *c*, *d*, *e*, *f* , *g*}].

1.12 SUMMARY

1. A set is a collection of well-defined and distinct objects. The objects are called members or elements of the set.
2. Sets are represented by capital letters and elements by small letters. If'a' is an element of set A,we write $a \in A$,otherwise $a \notin A$.
3. Sets are represented by (i) Roster or Tabular method and (ii)Ruleor Set-builder method.
4. Aset having no element is called empty set or null set or void set, denoted by$.
5. A set having a finite number of elements is called a finite set,otherwise it is called an infinite set.
6. Two sets AandB are equal,i.e. A=B if and only if every element of A is an element of B and also every element of B is an element of A, otherwise A∈B.
7. A is a subset of B, denoted by A⊆B if every element of Aisa element of B and A is a proper subset of B if A⊆B and A∈B. In this case, we write A⊂B.
8. A=B if and only if A⊂B and B⊂A.
9. The set of all the subsets 8 a set A is called the power set of A, denoted by P(A)or 2^A. If |A|=n, then $|P(A)|=2^n$.
10. Venn diagrams are plane geometrical diagrams used for representing relationships between sets.
11. The union of two sets AandB is A∪B which consists of all elements which are either in Aor B or in both. A∪B={x:xcAor xcB}
12. The intersection of two sets AandB is A∩Bwhich consists of all the elements common to both A andB.
13. For any two sets A and B, the difference set, A – B consists of all elements which are in A,but not in B.A– B={x: xcA and xøB}
14. The Universal Set U is that set which contains all the sets under any

    particular discussion as its subsets.

15. The complement of a set A, denoted by A^c or A′ is that set which consists of all those elements in U which are not in A.

A'={x:xcUandxøA}=U–A

16. Following are the Laws of Algebra of Sets: $A\cup A=A, A\cap A=A$

$A\cup B = B\cup A,\ A\cap B = B\cap A$

$A\cup(B\cup C) = (A\cup B)\cup C,\ A\cap(B\cap C) = (A\cap B)\cap C$

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C),\ A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

A∪$= A, A∪U = U, A∩U = A, A∩$= $

A∪A'=U,A∩A'=$,(A')'=A,U'=$,$'=U

$(A\cup B)'=A'\cap B', (A\cap B)'=A'\cup B'$.

1.13 KEYWORDS

Singleton Set: -If a set contains only one element it is called to be a singleton set.

Finite Set: -A set consisting of a natural number of objects, i.e. in which number element is finite is said to be a finite set. Consider the sets.

Infinite set: -If the number of elements in a set is finite, the set is said to be an infinite set.

Subset: -A subset A is said to be subset of B if every elements which belongs to A also belongs to B.

Proper set: -A set is said to be a proper subset of B if A is a subset of B, A is not equal to B or A is a subset of B but B contains at least one element which does not belong to A.

Power set: -Power set of a set is defined as a set of every possible subset. If the cardinality of A is n than Cardinality of power set is 2^n as every element has two options either to belong to a subset or not.

Universal set: -Any set which is a superset of all the sets under consideration is said to be universal set and is either denoted by omega or S or U.

Partition of set:A *partition* of *S* is a subdivision of *S* into nonoverlapping, nonempty subsets.

Multi sets: multi set isa collection of unordered numbers, where every element x occurs a finite number of times.

1.14 SELF ASSESSMENT TEST

1. Write'true'or'false'with proper justification:

   (i) the set of even prime numbers is an empty set

   (ii) {x : x+2 = 5, x < 0} is an empty set

   (iii) {3} c {1, 2, 3}

   (iv) x c{{x, y}}

   (v) {a, b} c {a, b, {c}}

   (vi) if A be any set, then $cA cU

2. Which of the following sets are equal? A = {x: x^2 + x – 2 =0}

   B = {x: x^2 – 3x + 2 = 0}

   C = {x : x ∈Z, |x| = 1} D = {–2, 1}

   E = {1, 2}

   F = {x: x^2 – 1 = 0}

3. Find the sets which are finite and which are infinite:

   (i) the set of natural numbers which are multiple of7

   (ii) the set of all districts of Assam

   (iii) the set of real numbers between 0 and1

   (iv) the set of lions in the world.

4. If U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {0, 2, 3, 6}, B = {1, 2, 6, 8},C={3,7,8,9},then findA′,B′,C′,$(AuB)nC$,$(AuB')uC'$, $(AnB)nC'$,$(A-C)uB'$,$(B-A)'nC$.

5. If $AuB = B$ and $AnB = B$,then what is the relation between A and B?

6. Verify the following identities with numerical examples:

   (i) $A - B = B' - A'$

   (ii) $(A - B)u(B - A) = (AuB) - (AnB)$

   (iii) $A - (BnC) = (A - B)u(A - C)$

   (iv) $A - (BuC) = (A - B)n(A - C)$

7. Write down the power set of the set $A = \{\{\$\}, a, \{b, c\}\}$.

8. Given $A = \{\{a, b\}, \{c\}, \{d, e, f\}\}$, how many elements are there in $P(A)$?

9. Using Venn diagrams show that

   (i) $AuBcAuC$but$B\o C$

   (ii) $AnBcAnC$but$B\o C$

   (iii) $AuB = AuC$butBsC.

10. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning (A), 5 had A and P,

12 had radio (R), 9 had A and R, 3 had all three options.

11 had power windows (W), 4 had R and W,

Find the number of cars that had: (a) onlyW; (b) onlyA; (c) onlyR; (d)R andW but notA; (e)A and R but not W; (f) only one of the options; (g) at least one option; (h) none of the options.

11. Let *S* = {1, 2, …, 8, 9}.Determine whether or not each of the following is a partition of *S*

(a) [{1, 3, 6}, {2, 8}, {5, 7, 9}] (c) [{2, 4, 5, 8}, {1, 9}, {3, 6, 7}]

(b) [{1, 5, 7}, {2, 4, 8, 9}, {3, 5, 6}] (d) [{1, 2, 7}, {3, 5}, {4, 6, 8, 9}, {3, 5}]

1.15 ANSWERS TO CHECK YOUR PROGRESS

Ans:1

(i) A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

(ii) B = {January, February, March, April, May, June, July, August, September, October, November, December}

(iii) C = {1, w,w^2}

(iv) D = {1, 2, 4, 5, 10, 20, 25, 50, 100}

(v) E = {A, B, E, G, L, R}

Ans: 2 B = C = E = F, A = D = G = H.

Ans:3A = {a, e, i, o, u}, B = D = {l, i, t, e},

C = {a, b, c, d, e}.

Ans: 4(a) C and E; (b) D and E; (c) A, B, and D; (d) None.

Ans : 5 (i) True, (ii) False, (iii) True, (iv) True, (v) False, (vi)True.

Ans:6 (a) $A \cap B$ = {2, 5}, $A \cap C$ = {1, 5};

(b) $A \cup B$ = {1, 2, 5, 6, 7}, $B \cup C$ = {1, 2, 3, 5, 7, 9};

(c) *A*C = {3, 4, 7, 8, 9}, CC = {2, 4, 6, 8};

(d) $A \backslash B$ = {1, 6}, $A \backslash C$ = {2, 6};

(e) $A\ \Phi B$ = {1, 6, 7}, $A\ \Phi\ C$ = {2, 3, 6, 7, 9};

(f) $(A \cup C) \backslash B$ = {1, 3, 6, 9}, $(B\ \Phi\ C) \backslash A$ = {3, 9}.

Ans:7 (i) ϕ, (ii) $, (iii) finite, (iv) infinite, (v) infinite, (vi) ϕ.

Ans:8 (1)B = {2, 3} =A

(2)A = {W, O, L, F}, B = {F, L, O, W} and so, A =B

(3)A⊂B; since b⊆A but b≠B.

Ans:9 A = {3, 6, 9, 12, 15, 18, 21, ...}, B = {1, 2, 3, ..., 18, 19, 20},

Hence A ∩B = {3, 6, 9, 12, 15, 18}

And B–A={1,2,4,5,7,8,10,11,13,14,16,17,19}

Ans10: *P(A)* has 25 = 32 elements as follows:

[Φ, {1}, {2}, {3}, {4}, {5}, {1, 2}, {1, 3}, {1, 4}, {1,5}, {2, 3}, {2, 4}, {2, 5},

{3, 4}, {3, 5}, {4, 5}, {1, 2,3}, {1, 2, 4}, {1, 2, 5}, {2, 3, 4}, {2, 3, 5}, {3, 4, 5},

{1, 3, 4}, {1, 3, 5}, {1, 4, 5}, {2, 4, 5}, {1, 2, 3, 4},{1, 2, 3, 5}, {1, 2, 4, 5},

{1, 3, 4, 5}, {2, 3, 4, 5}, *A*]

Ans11: (a) We want to find $n(N \cup T \cup F)$.

$$n(N \cup T \cup F)= n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

(b) 28+ 18 + 10 = 56 read exactly one of the magazines.

Ans12: (a) A has three elements, the sets {1, 2, 3}, {4, 5}, and {6, 7, 8}.

(b) n(A) = 3.

Ans:13 (*a*) *P*1 is not a partition of *S* since $f \in S$ does not belong to any of the cells.

(*b*) *P*2 is not a partition of *S* since $e \in S$ belongs to two of the cells.

(*c*) *P*3 is a partition of *S* since each element in *S* belongs to exactly one cell.

(*d*) *P*4 is a partition of *S* into one cell, *S* itself.

1.12REFERENCES/SUGGESTED READINGS

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
Course Code: MCA- 25	**AUTHOR: KAPILA DEVI**
Lesson No. 2	
RELATION AND FUNCTION	
REVISED /UPDATED SLM BY RENU BANSAL	

STRUCTURE

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know:

1. The concept of a relation
2. Different types of relation
3. equivalence relation
4. Partial ordered relation
5. Equivalence class
6. The concept of a function
7. Different types of function.
8. Composition of function

2.2 INTRODUCTION

Set theory may be called the language of modern mathematics. We know that a set is a well-defined collection of objects. Also we know the notion of subset of a set. If every element of a set B is in set A, then B is a subset of A. Symbolically we denote it by $B \subseteq A$.

Also we note that if A is a finite set having n elements, the number of subsets of A is 2^n. A gain if A, B are two non-empty sets,

Cartesian product

The Cartesian product of A and B is denoted by $A \times B$ and is defined by $A \times B = \{(a,b) : a \in A, b \in B\}$

(a, b) is called an ordered pair. Let $a \in A$, $c \in A$; $b \in B$, $d \in B$.

$\therefore$ (a, b)$\in$A x B, (c, d)$\in$A x B

We know that (a, b) = (c, d) $\Leftrightarrow$a=c, b=d.

Also we know that if A, B are finite and n(A)=x, n(B)= y, then n(AXB) = n(BXA)=xy

[Here n(A) denotes the number of elements of A]

If one of the sets A and B is infinite, then A X B and B X A are infinite.

In this unit we will study relations and functions which are subsets of Cartesian product of two sets.We will denote the set of natural numbers by 'N', the set of integers by Z, the set of rational numbers by Q, the set of real numbers by IR, the set of complex numbers by C.

2.3 CONCEPT OFRELATION

Let us consider the following sentences.

- 11 is greater than10.
- 35 is divisible by7.
- New Delhi is the capital of India.

In each of the sentences there is a relation between two 'objects'. Now let us see what is meant by relation in set theory.

Definition Let A and B be two non-empty sets. A subset R of AXB is said to be a **relation** from Ato B.If A=B, then any subset of A X A is said to be a relation on A. If R $\subseteq$ AxB, and (a,b)$\in$R;a$\in$A,b$\in$B,it is also written as a Rb and is read as 'a is R related to b'.

Note. The set of the first components of the ordered pairs of R is called the domain and the set of the second components of the ordered pairs of R is called the range of R. If A,B are finite sets and n(A)=x,n(B)=y; then n(AXB)=xy.So,the number of subsets of AXB is2^{xy}.Therefore, the number of relations from A to B is 2^{xy}.

Following examples shows representation of relations.

Example 1: Let A ={1, 2, 3},B={8, 9}

$\therefore$ A x B ={(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)}

Let R = {(1, 8), (2, 9), (3, 9)}

Clearly $R \subseteq AxB$

$\therefore$ R is a relation from A to B. Here 1R8, 2R9, 3R9

Domain of R = {1, 2, 3}

Range of R = {8, 9}

Example 2: Let IR be the set of real numbers. Let R = {(x, y): x, y $\in$IR, x<y} $\subseteq$IR x IR

$\therefore$ R is a relation of IR.

3<5, $\therefore$(3, 5)$\in$R i.e. 3R5

19<27, $\therefore$(19, 27)$\in$R i.e. 19R27 But 5>3, $\therefore$(5, 3)$\notin$R, i.e. 5 R/3.

Example 3: Let X be the set of odd integers. Let R = {(x, y): x, y $\in$X and x+y is odd}

We know that the sum of two odd integers is an even integer.

$\therefore$ if x, y are odd, then x+y cannot be odd.

$\therefore$ R =$\phi \subseteq$X x X

In this case R is called a null relation on X.

Example 4: Let E be the set of even integers. Let R = {(x, y): x, y $\in$ E and x+y is even}

We know that the sum of the even integers is an even integer.

$\therefore$ if x and y are even, then x+y is always even.

$\therefore$ R = E x E $\subseteq$E x E

In this case R is called a universal relation on E.

2.3.1. IDENTITY RELATION

Let A be a non-emptyset. I = {(a, a): a$\in$A} $\subseteq$A xA $I_A$is called the identity relation on A.

Example 5: Let A = {1, 2, 3, 4} Then I = {(1, 1), (2, 2), (3, 3), (4, 4)} ⊆A x A.

Clearly I_A is the identity relation on A.

2.3.2. INVERSE RELATION

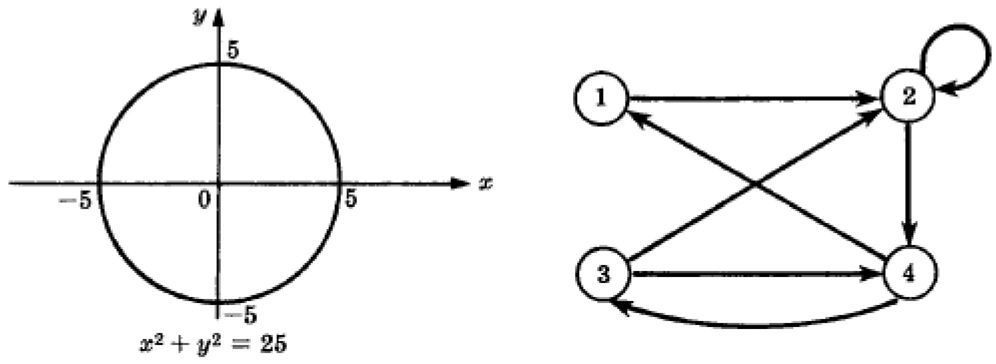
Let A,B be two non-empty sets.Let R be a relation from AtoB,i.e. R⊆AxB. The inverse relation of R is denoted by R^{-1},and is defined by R^{-1}= {(b, a): (a, b)∈R} ⊆B xA. Clearly, domain of R^{-1}= range of R range of R^{-1}= domain of R

2.4 REPRESENTATION OF RELATION

There are various ways of picturing relations.

Let *S* be a relation on the set **R** of real numbers; that is, *S* is a subset of **R**2 = **R** × **R**. Frequently, *S* consists of all ordered pairs of real numbers which satisfy some given equation *E(x, y)* = 0 (such as *x*2 + *y*2 = 25).

Since **R**2 can be represented by the set of points in the plane, we can picture *S* by emphasizing those points in the plane which belong to *S*. The pictorial representation of the relation is sometimes called the graph of the relation. For example, the graph of the relation *x*2+*y*2 = 25 is a circle having its center at the origin and radius 5.



(a) (b)

Directed Graphs of Relations on Sets

There is an important way of picturing a relation *R* on a finite set. First we write down the elements of the set, and then we draw an arrow from each element *x* to each element *y* whenever *x* is related to *y*. This diagram is called the directed graphof the relation. For example, shows the directed graph of the following relation *R* on the set *A* = {1, 2, 3, 4}:

R = {(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)}

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under *R*.

Pictures of Relations on Finite Sets

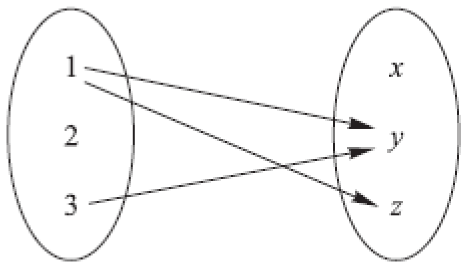
Suppose *A* and *B* are finite sets. There are two ways of picturing a relation *R* from *A* to *B*.

(i) Form a rectangular array (matrix) whose rows are labeled by the elements of *A* and whose columns are labeled by the elements of *B*. Put a 1 or 0 in each position of the array according as *a* ∈*A* is or is not related to *b* ∈*B*. This array is called the *matrix of the relation*.

(ii) Write down the elements of *A* and the elements of *B* in two disjoint disks, and then draw an arrow from *a* ∈*A* to *b* ∈*B* whenever *a* is related to *b*. This picture will be called the *arrow diagram* of the relation. Figure below pictures the relation *R* by the above two ways.

	x	*y*	*z*
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

2.5 TYPES OF RELATION

Let A be a non-empty set, and R be a relation on A, i.e. R ⊆A x A.

1. R is called reflexive if(a,a)∈R,i.e;aRa, for all a∈A.

2. R is called symmetric if whenever (a, b)∈R, then (b, a)∈R, i.e, if whenever aRb,then bRa;

$a, b \in A$.

3. R is called anti-symmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$, i.e., if aRb, bRa $\Rightarrow a=b$; $a, b \in A$.

4. R is called transitive if whenever $(a,b),(b,c) \in R$, then $(a,c) \in R$, i.e., if whenever aRb, bRc, then aRc; $a,b,c \in A$.

Example 7: Let A = {1, 2, 3} R = (1, 1), (2, 2), (1, 2), (2, 1)} Examine if R is reflexive, symmetric, anti-symmetric, transitive.

Solution: Here $(1, 1) \in R$, $(2, 2) \in R$; but $(3, 3) \notin R$. $\therefore$ R is not reflexive.

Again, $(a, b) \in R \Rightarrow (b, a) \in R$ $\therefore$ R is symmetric.

Again, $(1, 2) \in R$, $(2, 1) \in R$, but $1 \neq 2$ $\therefore$ R is not anti-symmetric.

Again, if $(a, b) \in R$, $(b, c) \in R$, then $(a, c) \in R$ $\therefore$ R is transitive.

Example8: Let Z be the set of integers, and $R=\{(x,y):x,y \in Z, x \leq y\}$ Examine if R is reflexive, symmetric, anti-symmetric and transitive.

Solution: We have $a \leq a$, $\forall a \in Z$ i.e, $(a, a) \in R$, $\forall a \in Z$

$\therefore$ R is reflexive. Again, if $a \leq b$, $b \not\leq$ i.e;$(a, b) \in R \not\Rightarrow (b,a) \in R$

$\therefore$ R is not symmetric.

Again, $a \leq b$, $b \leq a \Rightarrow a = b$

i.e. $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$

$\therefore$ R is anti-symmetric.

Again $a \leq b$, $b \leq c \Rightarrow a \leq c$

i.e. $(a, b), (b, c), \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Example 9: Give an example of a relation which is transitive, but neither reflexive nor symmetric.

Solution: LetA={1,2} Let R = {(1, 1), (1, 2)} $\subseteq$A x A Clearly R is a relation on A. Here (2, 2)$\notin$R. $\therefore$R is not reflexive.

Again (1, 2) $\in$R, but (2, 1) $\notin$R $\therefore$R is not symmetric. But R is transitive.

2.5.1 EQUIVALENCERELATION

Let A be a non-empty set. A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive.

Example10: Let Z be the set of integers and R = {(x, y): x, y $\in$ Z and x + y is even} Examine if R is an equivalence relation on Z.

Solution: Let x $\in$Z

$\therefore$x + x is even

$\Rightarrow$(x, x) $\in$R, $\forall$x$\in$Z

$\therefore$R is reflexive.

(x, y)$\in$R $\Rightarrow$x + y is even

$\Rightarrow$y + x is even

$\Rightarrow$ (y, x) $\in$ R.

$\therefore$R is symmetric. (x, y) $\in$R, (y, z) $\in$R

$\Rightarrow$x + y is even, y + z is even

$\Rightarrow$(x + y) + (y + z) is even

$\Rightarrow$(x+ z) + 2y is even

$\Rightarrow$x + z is even (x, z) $\in$R

$\therefore$R is transitive

$\therefore$R is an equivalence relation on Z.

Example11: Let A be the set of all straight lines in a plane. Let R={(x,y):x,y $\in$ Aandx$\perp$y} Examine if R is reflexive, symmetric and transitive.

Solution:A line cannot be perpendicular to itself,i.e.x $\not\perp$ x

$\therefore$(x, x) $\notin$R

$\therefore$ R is not reflexive

If a line x is perpendicular to an other line y, then y is perpendicular to x,i.e. x$\perp$y $\Rightarrow$y$\perp$x

$\therefore$(x, y) $\in$ R $\Rightarrow$ (y, x) $\in$ R

$\therefore$R is symmetric.

Ifx$\perp$y,y$\perp$z,thenx $\not\perp$z

$\therefore$(x,y),(y,z)$\in$R $\not\Rightarrow$(x,z)$\in$R

If x is perpendicular to y, y is perpendicular to z, then x is parallel to z.

$\therefore$R is not transitive.

Example12: Let A be the set of all straight lines in a plane. Let R={(x,y):x,y$\in$Aandx$\parallel$y}. Examine if R is an equivalence relation on A.

Solution: A line is parallel to itself, i.e. x$\parallel$x $\forall$x $\in$A

$\therefore$(x, x) $\in$R, $\forall$x $\in$R

$\therefore$ R is reflexive. x$\parallel$y $\Rightarrow$y$\parallel$x

$\therefore$(x, y) $\in$R $\Rightarrow$(y, x) $\in$ R

$\therefore$R is symmetric. x$\parallel$y, y$\parallel$z $\Rightarrow$ x$\parallel$z

$\therefore$(x, y), (y, z) $\in$R $\Rightarrow$ (x, z) $\in$ R.

$\therefore$ R is transitive

Thus, R is an equivalence relation on A.

Example13: Let IN be the set of natural numbers. Let a relation R be defined on INxIN by (a,b) R (c,d) if and only if ad=bc Show that R is an equivalence relation on IN x IN.

Solution: : We have ab = ba

$\therefore$(a, b) R (a, b)

$\therefore$R is reflexive

(a, b) R (c, d) $\Rightarrow$ad = bc; a, b, c, d, $\in$IN

$\Rightarrow$cb = da

$\Rightarrow$(c, d) R (a, b)

$\therefore$R is symmetric

(a, b) R (c, d) and (c, d) R (e, f); a, b, c, d, e, f, $\in$IN

$\Rightarrow$ad = bc and cf = de

$\Rightarrow$ adcf = bcde

$\Rightarrow$af = be

$\Rightarrow$(a, b) R (e, f)

$\therefore$R is transitive.

Thus, R is an equivalence relation on IN x IN.

Example 14: (Congruence modulo n) Let Z be the set of integers, and n be any fixed positive integer. Let a, b$\in$Z a is said to be congruent to b modulo n if and only if a–b is divisible by n. Symbolically, we write. a $\equiv$ b (mod n) Show that the relation 'congruence modulon' is an equivalence relation on Z.

Solution: We know that a–a is divisible by n, i.e., a $\equiv$ a (mod n)

$\therefore$the relation is reflexive. Let a $\equiv$ b (mod n)

$\Rightarrow$ a–b is divisible by n.

$\Rightarrow$b–a is divisible by n.

$\Rightarrow b \equiv a \pmod{n}$

$\therefore$ the relation is symmetric.

Let $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$

$\Rightarrow a–b$ is divisible by n, $b–c$ is divisible by n

$\Rightarrow a–b+b–c$ is divisible by n

$\Rightarrow a–c$ is divisible by n

$\Rightarrow a \equiv c \pmod{n}$

$\therefore$ the relation is transitive.

Thus, the relation 'congruence modulo n' is an equivalence relation on Z.

We know that 15-3 is divisible by 4.

$\therefore$ 15 is congruent to 3 modulo 4i.e.

$15 \equiv 3 \pmod{4}$

15–3 is not divisible by 7

$\therefore$ 15 is not congruent to 3 modulo 7

i.e.$15 \not\equiv 3 (\text{mod} 7)$

Example 15: Let Z be the set of integers. Let R = {(a, b): a, b $\in$Z, ab$\geq$ 0}

Examine if R is an equivalence relation on Z.

Solution: We have aa $\geq$ 0

$\therefore$(a, a)$\in$R, $\forall$ a$\in$Z

$\therefore$R is reflexive Let (a, b) $\in$ R

$\therefore$ab $\geq$0

$\Rightarrow$ ba $\geq$0

$\Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

We have(–2) x 0 = 0, 0 x2=0

$\therefore (-2, 0), (0, 2) \in R$

But (–2) x 2 = –4<0

$\therefore (-2, 2) \notin R$

$\therefore$ R is not transitive.

Thus, R is not an equivalence relation.

Theorem 1: The inverse of an equivalence relation is also an equivalence relation.

Proof. Let A be a non-empty set.Let R be an equivalence relation on A R is reflexive.

$\therefore (x, x) \in R, \forall x \in A$

$\Rightarrow (x, x) \in R^{-1}, \forall x \in A$

$\therefore R^{-1}$ is reflexive. Let $(x, y) \in R^{-1}$

This $\Rightarrow$ $(y, x) \in R$ [by def^n of R^{-1}]

$\Rightarrow (x, y) \in R$ [$\therefore$ R issymmetric]

$\Rightarrow (y, x) \in R^{-1}$

$\therefore R^{-1}$ issymmetric.

Let $(x, y), (y,z) \in R^{-1}$

$\Rightarrow (y, x), (z, y) \in R$ [by def^n of R^{-1}]

$\Rightarrow (z, y), (y, x) \in R$

$\Rightarrow (z, x) \in R$ [$\therefore$ R is transitive]

$\Rightarrow (x, z) \in R^{-1}$

$\therefore R^{-1}$ is transitive

Thus, R^{-1} is an equivalence relation.

Theorem 2: The intersection of two equivalence relations is also an equivalence relation.

Proof. Let A be a non-empty set.

Let R and S be two equivalence relations on A. R and S are reflexive

$\therefore (x, x) \in R$ and $(x, x) \in S, \forall x \in A$

$\Rightarrow (x, x) \in R \cap S, \forall x \in A$

$\therefore R \cap S$ is reflexive. Let $(x, y) \in R \cap S$

This $\Rightarrow (x, y) \in R$ and $(x, y) \in S$

$\Rightarrow (y, x) \in R$ and $(y, x) \in S$ [$\therefore$ R, S are symmetric]

$\Rightarrow (y, x) \in R \cap S$

$\therefore R \cap S$ is symmetric.

Let $(x, y) \in R \cap S, (y, z) \in R \cap S$

$\Rightarrow (x, y) \in R$ and $(x, y) \in S$, $(y, z) \in R$ and $(y, z) \in S$

$\Rightarrow (x, z) \in R$ and $(x, z) \in S$ [$\therefore$ R, S are transitive]

$\Rightarrow (x, z) \in R \cap S$

$\therefore R \cap S$ is transitive.

Thus, $R \cap S$ is an equivalence relation.

Remarks: The union of two equivalence relations is not necessarily an equivalence relation.

Let us consider A = {1, 2, 3}

Now, R = {(1, 1), (2, 2), (3, 3), (1, 3), (3,1)}

S = {(1, 1), (2, 2), (3, 3), (1, 2), (2,1)}

are two equivalence relations on A.

$R \cup S$ = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (1, 2), (2, 1)}

(2, 1) $\in R \cup S$, (1, 3)$\in R \cup S$)

But (2, 3) $\notin R \cup S$

$\therefore R \cup S$ is not transitive

$\therefore R \cup S$ is not an equivalence relation.

2.5.2 PARTIAL ORDERING RELATIONS(POSETS)

Suppose *R* is a relation on a set *S* satisfying the following three properties:

1. (Reflexive) For any $a \in S$, we have *aRa*.
2. (Antisymmetric) If *aRb* and *bRa*, then *a* = *b*.
3. (Transitive) If *aRb* and *bRc*, then *aRc*.

Then *R* is called a *partial order* or, simply an *order* relation, and *R* is said to define a *partial ordering* of *S*. The set *S* with the partial order is called a **partially ordered set**or, simply, an *ordered set* or **poset**. We write *(S,R)*

when we want to specify the relation *R*.

The most familiar order relation, called the *usual order*, is the relation ≤ (read “less than or equal”) on the positive integers **N** or, more generally, on any subset of the real numbers **R**.

here we give some examples.

EXAMPLE :

(a) The relation ⊆ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set *A*.

(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) The relation ≤ on the set **R** of real numbers is reflexive, antisymmetric, and transitive. Thus ≤ is a partial ordering on **R**.

(c) The relation "*a* divides *b*," written $a \mid b$, is a partial ordering on the set **N** of positive integers. However, "*a* divides *b*" is not a partial ordering on the set **Z** of integers since $a \mid b$ and $b \mid a$ need not imply $a = b$. For

example, 3|−3 and −3 | 3 but 3 _= −3.

EXAMPLE:

Consider the set **Z** of integers. Define *aRb* by $b = a^r$ for some positive integer *r*. Show that *R* is a partial order on **Z**, that is, show that *R* is: (*a*) reflexive; (*b*) antisymmetric; (*c*) transitive.

(a) *R* is reflexive since $a = a^1$.

(*b*) Suppose *aRb* and *bRa*, say $b = a^r$ and $a = b^s$. Then $a = (a^r)^s = a^{rs}$. There are three possibilities:

(i) $rs = 1$,

(ii) $a = 1$, and

(iii) $a = -1$. If $rs = 1$ then $r = 1$ and $s = 1$ and so $a = b$. If $a = 1$ then $b = 1^r = 1 = a$, and,

similarly, if $b = 1$ then $a = 1$. Lastly, if $a = -1$ then $b = -1$ (since b _= 1) and $a = b$. In all three cases, $a = b$.

Thus *R* is antisymmetric.

(c) Suppose *aRb* and *bRc* say $b = a^r$ and $c = b^s$. Then $c = (a^r)^s = a^{rs}$ and, therefore, *aRc*. Hence *R* is transitive.

Accordingly, because given relation satisfy all the three relations ,so *R* is a partial order on **Z**

2.6 EQUIVALENCECLASS

Let us consider the set Z of integers

Let R = {(a, b): a, b $\in$Z, a–b is divisible by 3}

i.e. R = {(a, b): a, b$\in$Z, a $\equiv$b (mod 3)}

Clearly R is reflexive, symmetric and transitive.

$\therefore$R is an equivalence relation on Z.

Let [0] denote the set of integers congruent to 0 modulo 3. Then [0] = {...., –9, –6, –3, 0, 3, 6, 9, ...}

Let [1] denote the set of integers congruent to 1 modulo 3. Then [1] = {..., –8, –5, –2, 1, 4, 7, 10, ...}

Let[2]denotethesetofintegerscongruentto2modulo3.Then [2]={....,–7,–4,–1,2,5,8,11, …}

We see that

...... = [0] = [3] = [6]=.....

...... = [1] = [4] = [7]=.....

...... = [2] = [5] = [8]=.....

Each of [0], [1], [2] is called an equivalence class. [0] is the equivalence class of 0, [1] is the equivalence class of 1, [2] is the equivalence class of 2.We see that there are 3 distinct equivalence classes, viz, [0], [1], [2].

Also we note that [0] U [1] U [2] = z, [0]$\cap$[1]=ϕ,[1]$\cap$[2]=ϕ[2]$\cap$[0] =ϕ

The set {[0], [1], [2]} is called a partition of Z.

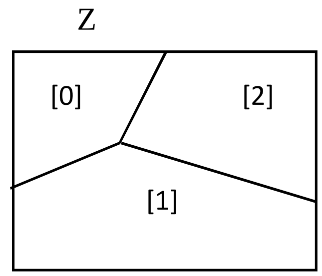


FIG 2.1

Definition: Let A be a non-empty set and R be a relation on A. For any a$\in$A, the equivalence class [a] of a is defined by [a] = {x$\in$A : xRa}i.e.the equivalence class [a] of a is the collection of all those elements of A which are related to a under the relation R.

Note: [a] $\neq \phi$, $\because$ aRa

Definition: Let A be a non-empty set.The set P of non-empty subsets of A is called a partition of A if

(i) A is the union of all members of P,

(ii) any two distinct members of P are disjoint.

Theorem 3: Let A be a non-empty set and R be an equivalence relation of A. Let a, b∈A. Then [a] = [b] if and only if (a, b) ∈R.

Proof. Let [a] = [b]

R is reflexive, aRa, a∈[a]

⇒a∈[b]   [[a]=[b]]

⇒aRb ⇒(a, b) ∈R

Conversely: Let (a,b)∈R. aRb

Let x∈[a]. xRa

Now, xRa and aRb

⇒ xRb   [R is transitive]

⇒x∈[b]

Thus, x ∈[a] ⇒x∈[b] [a]⊆[b]   ...(1)

Again,lety∈[b] yRb

Now, yRb and bRa   [R is symmetric, aRb ⇒bRa]

yRa   [R is transitive]

⇒y∈[a]

Thus, y∈[b] ⇒y∈[a] [b]⊆[a]   ...(2)

From (1)and(2),   [a] =[b]

Theorem 4: Two equivalence classes are either equal or disjoint.

Proof. Let A be a non-empty set. Let R be an equivalence relation onA.

Let a, b∈A.

Then [a], [b] are either not disjoint or disjoint. Let [a], [b] be not disjoint,

i.e. $[a] \cap [b] \neq \phi$.

Let $x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow$xRa and xRb

$\Rightarrow$aRxandxRb [R is symmetric, xRa$\Rightarrow$aRx]

$\Rightarrow$aRb [R is transitive]

Let$y \in [a]$

$\Rightarrow$yRa

Now yRa and aRb$\Rightarrow$yRb [R is transitive]

$\Rightarrow y \in [b]$

Thus, $y \in [a] \Rightarrow y \in [b]$

$[a] \subseteq [b]$Similarly, $[b] \subseteq [a]$

$[a] = [b]$

So, if [a], [b] are not disjoint, they are equal. [a], [b] are either equal or disjoint.

2.7 CONCEPT OF FUNCTION

Let A and B be two non-empty sets, and $f \subseteq A \times B$ such that

(i) $(x,y) \in f, \forall x \in A$ and any $y \in B$

(ii) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$.

In this case f is said to be a function (or a mapping) from the set A to the set B. Symbolically we write it as $f:A \rightarrow B$.

Here A is called the domain and B is called the codomain of f.

Example 16: Let A = {1, 2}, B = {7, 8, 9}

f = {(1, 8), 2, 7)} $\subseteq$A x B.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f.

$\therefore$ f is a function from A to B.

Example 17: Let A = {1, 2}, B = {7, 8, 9}

g = {(1, 7), (1, 9)} $\subseteq$A x B

Here, two distinct ordered pairs have the same first component.

$\therefore$ f is not a function from A to B.

Example 18: Let A = {1, 2, 3, 4}, B = {x, y, z, w} Are the following relations from A to B be functions?

(i) f_1 = {(1, x), (1, w), (2, x), (2, z), (4, w)}

(ii) f_2 = {(1, y), (2, z), (3, x), (4, w)}

Solution: (i) No. Here two distinct ordered pairs (1, x), (1, w) have the same first component.

(ii) Yes. Here, each element of A appears as the first component exactly in one of the ordered pairs of f_2. Thus we see that.

Every function is a relation, but every relation is not a function.

We observe that if A and B are two non- empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is

made, is called a function from the set A to the set B.The rules are denoted by f,g etc. The sets A, B may be the same.

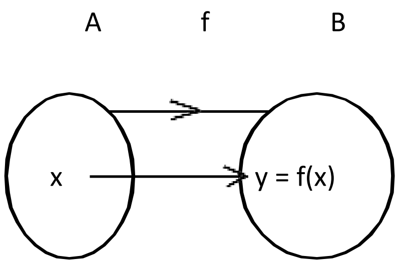


FIG 2.2

Let f be a function from A to B i.e

f:A→B.The unique element y of B that is associated with x of A is called the image of x under f.Symbolically we write it as f = f(x). x is called the pre- image of y.The set of all the images under f is called the range off.

Example 19: Let IN be the set of natural members, and Z be the set of integer and f : IN → Z, f(x) = $(–1)^x$; x ∈IN. Clearly, domain of f = IN

codomain of f = Z

Now f(1) = $(–1)^1$ = –1, f(2) = $(–1)^2$ = 1, f(3) = $(–1)^3$ = –1 and so on.

range of f = {–1, 1}

Identity Function: Let A be a non-empty set and i:A→A,i(x)=x,∀x∈A I is called the identity function.

Note: In case of identity function, domain and co domain are the same.

Constant Function: Let A,B be two non-empty sets and f:A→B be a function such that f(x) = k, ∀x ∈A

f is called a constant function.

Note: The range of a constant function is a singleton set.

2.8 TYPES OF FUNCTION

Let A, B be two non-empty sets and f: A B be a function.

If there is at least one element in B which is not the image of any element in A, then f is called an “into” function.

If each element in B is the image of at least one element in A,then f is called an“onto”function (or a surjective function or a surjective).

Note: In case of an onto function, range of f=co domain of f.

If different elements in A have different images in B,then f is called a one-one function(or an injective function or an injection).

If two (or more) different elements in A have the same image in B,then f is called a many-one function.

A function is said to be bijective if it is one-one (injective) and onto (surjective).

Note: Identity function is a bijective function.

How to prove that f is one-one?

Let $x_1,x_2, \in A$

If $f(x) = f(x_2) \Rightarrow x_1 = x_2$, then f is one-one. or,

If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, then f is one-one.

How to prove that f is onto?

Let $y \in B$ (codomain) Let $y = f(x)$

We find x in terms of y

If $x \in A$, then f is onto; otherwise not.

Example22: LetA={1,2,3,}.Write down all the bijective function from A to itself.

Solution:

$i: A \to A$, $i(1) = 1$, $i(2) = 2$, $i(3) = 3$

$f_1: A \to A$, $f_1(1) = 1$, $f_1(2) = 3$, $f_1(3) = 2$

$f_2: A \to A$, $f_2(1) = 2$, $f_2(2) = 1$, $f_2(3) = 3$

$f_3: A \to A$, $f_3(1) = 3$, $f_3(2) = 2$, $f_3(3) = 1$

f_4: A→A, $f_4(1) = 2$, $f_4(2) = 3$, $f_4(3) = 1$

f_5: A→A, $f_5(1) = 3$, $f_5(2) = 1$, $f_5(3) = 2$

Note. There are 6 =3! bijective functions from A = {1, 2, 3} to itself.

If A has n elements, there are n! bijective functions from A to itself.

Theorem5: Let A be a finite set,andf:A→Abe onto.Then f is one-one.

Proof. Let A be a finite set having n elements.

Let A = $\{a_1, a_2, ..., a_n)$, where a_i' s are distinct.

Let f : A→A be onto. Now, range of f

= $\{f(a_1), f(a_2), ..., f(a_n)\}$f is onto,

range of f = codomain of f =A. A =$\{f(a_1),f(a_2),$ $,f(a_n)\}$

A has n elements,

$f(a_1),f(a_2),$ .... , $f(a_n)$ are distinct.

Thus, distinct elements in A (domain) have distinct images in A(codomain). f is one-one.

Note: The result does not hold good if A is an infinite set. Let IN be the set of natural numbers.

Let f : IN→IN, f(x) =1, if x = 1 and x = 2,

= x–1, if x ≥ 3 Clearly, f is onto, but not one-one.

Theorem 6: Let A be a finite set, and f: A→A be one-one. Then f is onto.

Proof: Let A be a finite set having n elements. Let A =$\{a_1,a_2,$ ……, $a_n\}$, where a_i's are distinct.

Let f : A→ A be one-one.

$f(a_1),f(a_2),$ ....,$f(a_n)$ are n distinct elements of A (codomain).

Let u ∈A (codomain).

Let u = $f(a_i)$, $1 \leq i \leq n$

there exists $a_i \in A$ (domain) such that $f(a_i) = u$. f is onto.

Note: The result does not hold good if A is an infinite set. Let IN be the set of natural numbers.

Let f: IN →IN, f(x) = 5xClearly, f is one-one, but not onto.

2.8.1 INVERSE FUNCTION

A function f: A → B is invertible if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

Theorem : A function f: A → B is invertible if and only if f is both one-to-one and onto.

If f : A → B is one-to-one and onto, then f is called a one-to-one correspondence between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

Example:

Let f:R→R be defined by f(x)=3x+1. Find the inverse function f^{-1}.

Solution: For any input x, the function machine corresponding to f spits out the value y=f(x)=3x+1. We want to find the function f^{-1}that takes the value y as an input and spits out x as the output. In other words, y=f(x) gives y as a function of x, and we want to find $x=f^{-1}(y)$ that will give us x as a function of y.

To calculate x as a function of y, we just take the expression y=3x+1 for y as a function of x and solve for x.

$$y=3x+1$$

$$y-1= 3x$$

$$(y-1)/3=x$$

Therefore, we found out that x=y/3−1/3 , so we can write the inverse function as

$$f^{-1}(y)=y/3-1/3.$$

In the definition of the function f^{-1}, there's nothing special about using the variable y. We could use any other variable, and write the answer as

$$f^{-1}(x)=x/3-1/3.$$

The placeholder variable used in the formula for a function doesn't matter.

2.9 Composition of function

Composition of Function

Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

That is, we find the image of a under f and then find the image of $f(a)$ under g. we use the functional notation $g \circ f$ for the composition of f and g .

Consider any function $f: A \to B$. Then

$$f \circ 1_A = f \text{ and } 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively.

Example:

Let A = {a, b, c},B = {x, y, z},C = {r, s, t}. Let f: A → B and g: B → C be defined by:

f = {(a, y)(b, x), (c, y)} and g = {(x, s), (y, t), (z, r)}.

Find: composition function $g \circ f: A \to C$;

Solution:

Use the definition of the composition function to compute:

$(g \circ f)(a) = g(f(a)) = g(y) = t$

$(g \circ f)(b) = g(f(b)) = g(x) = s$

$(g \circ f)(c) = g(f(c)) = g(y) = t$

That is $g \circ f$ = {(a, t), (b, s), (c, t)}.

2.10 CHECK YOUR PROGRESS

1. Let A, B be two finite sets, and n(A)=4, n(B)=3. How many relations are there from A to B?
2. Let A = {1, 2, 3, 4, 5} Write down the identity relation on A.
3. Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Find $S \times T \times W$.

4. Let A = {4, 5, 6}, B = {7, 8, 9} Let R = {(4, 7), (5, 8), (6, 7), (6, 8), (6, 9)}. Write down R^{-1}.
5. Each of the following defines a relation on the positive integers **N**:

   (1) "*x* is greater than *y*." (3) $x + y = 10$

   (2) "*xy* is the square of an integer." (4) $x + 4y = 10$.

   Determine which of the relations are: (a) reflexive; (b) symmetric; (c) antisymmetric; (d) transitive.

6. Let $S = \{1, 2, 3, . . . , 9\}$, and let ∼be the relation on $A \times A$ defined by

$(a, b) \sim (c, d)$ whenever $a + d = b + c$.

   (a) Prove that ∼ is an equivalence relation.

   (b) Find [(2, 5)], that is, the equivalence class of (2, 5).

7. Let A be the set of all triangles in a plane. Let R = {(x, y): x, y ∈A., x is similar to y} Examine if R is an equivalence relation on A.
8. Let IN be the set of n natural numbers. Let f: IN →IN, $f(x) = x^2+1$ Examine if f is (i) one-one, (ii) onto.
9. Let $V = \{1, 2, 3, 4\}$. For the following functions $f: V \to V$ and $g: V \to V$, find:(a) $f \circ g$; (b), $g \circ f$; (c) $f \circ f$:

f = {(1, 3), (2, 1), (3, 4), (4, 3)} and g = {(1, 2), (2, 3), (3, 1), (4, 1)}

10.Decide which of the following functions are: (a) one-to-one; (b) onto; (c) both; (d) neither.

(1) f: **Z**2 → **Z** where f (n, m) = n − m; (3) h: **Z** × (**Z**\0) → **Q** where h(n, m) = n/m;

(2) g: **Z**2 → **Z**2 where g(n, m) = (m, n); (4) k: **Z** → **Z**2 where k(n) = (n, n).

11. Let *l* be any collection of sets. Is the relation of set inclusion ⊆ a partial order on *l* ?

2.11 SUMMARY

1. Empty set, I_A={(a,a):a∈A} is called the identity relation on A.
2. If A,B are two non-empty sets, a subset of AXB in said to be a relation from AtoB.
3. If A,B are two finite sets and n(A)=x, n(B)=y, the number of relations from A to B is 2^{xy}.
4. If A is a non-n-empty and R is a relation from A to B, the inverse relation $R^{-1}$is defined by R^{-1}= {(b, a): (a, b)∈R}, which is a relation from B to A.
5. A relation R on a non-empty set A is called.
   a. Reflexive if (a,a) ∈R, for all a∈A;
   b. symmetric if whenever (a, b) ∈R, (b, a) ∈R;
   c. anti-symmetric if (a, b) ∈R, (b, a) ∈R ⇒a =b;
   d. transitive if whenever (a,b),(b,c)∈R,then(a,c)∈R.
6. A relation R on a non-empty set A is called an equivalence relation if it is reflexive, symmetric and transitive.
7. The inverse of an equivalence relation is also an equivalence relation.
8. The intersection of two equivalence relations is also an equivalence relation.
9. If R is a relation on a non-empty set A, then for any a∈A; the equivalence class [a] of a is the

collection of all those elements of A which are related to a under the relation R.

10. Two equivalence classes are either equal or disjoint.
11. If A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from AtoB.
12. Every function is a relation, but every relation is not a function.
13. If different elements in domain have different images in codomain, then the function is one-one(injective).
14. If each element in codomain is the image of at least one element in domain then the function is onto(surjective).
15. A function is bijective if it is injective and surjective.
16. If A is a finite set and f: A→A is onto, then f is one-one.
17. If A is a finite set and f:A→A is one-one, then f is onto.

2.12 KEYWORDS

Relation: A binary relation R from set x to y (written as xRy or R(x,y)) is a subset of the Cartesian product x×y. If the ordered pair of G is reversed, the relation also changes. Generally, an n-ary relation R between sets A1…, and An is a subset of the n-ary product A1×⋯×An. The minimum cardinality of a relation R is Zero and maximum is n^2 in this case.

Equivalence Relation: A relation is an equivalence Relation if it is reflexive, symmetric, and transitive

Function: A function or mapping (Defined as f: X→Y) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Injective / One-to-one function: A function f: A→B is injective or one-to-one function if for every b∈B, there exists at most one a ∈A such that f(s)=t.

Surjective / Onto function: A function f: A→B is surjective (onto) if the image of f equals its range. Equivalently, for every b∈B, there exists some a∈A such that f(a)=b. This means that for any y in B, there exists some x in A such that y=f(x).

Bijective / One-to-one Correspondent: A function f: A→B is bijective or one-to-one correspondent if and only if f is both injective and surjective

2.13 SELF ASSESSMENT TEST

1. Let A = {1, 2, 3}, B = {3, 4, 5}. How many relations are there from A to B? Write down any four relations from A to B.

2. Let A = {3, 4, 5}, B = {5, 6, 7} Which of the following relations are functions from A to B? If it is a function, determine whether it is one-one and whether it is onto?

   (i) {(3, 5), (4, 5), (5, 7)}

   (ii) {(3, 7), (4, 5), (5, 6)}

3. Let Q be the set of rational numbers and f:Q→Q be a function defined f(x)=4x+5. Examine if f is bijective

4. Let A be the set of all triangles in a plane. Let R = {(x, y): x, y ∈A and x is congruent to y} Examine if R is an equivalence relation on A.

5. Let IN be the set of natural numbers. A relation R is defined on INxIN by (a,b), R(c,d) if and only if a+d=b+c. Show that R is an equivalence relation on IN x IN.

6. Let A = {1, 2, 3} R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} Show that R is reflexive but neither symmetric nor transitive.

7. Let C be the set of complex numbers and IR be the set of real numbers. Let f : C→IR, f(z) = |z|, z ∈C. Examine if f is (1) one-one, (ii) onto.

8. Let A, B be two non-empty sets. Let f:AxB→B xA, f (a,b)=(b,a);(a,b) ∈AxB Show that f is bijective.

9. Let f:INxIN, f(a,b)=$3^a 4^b$,(a,b) ∈INxIN Examine if f is(i)one-one,(ii)onto.

10. Let f : Z x Z →Z, f(a, b) = ab, (a, b) ∈Z x Z Examine if f is (i) injective, (ii)surjective.

11. Let V = {1, 2, 3, 4}. For the following functions f: V → V and g: V → V , find:

(a) f ∘g; (b), g∘f ; (c) f ∘f :

f = {(1, 3), (2, 1), (3, 4), (4, 3)} and g = {(1, 2), (2, 3), (3, 1), (4, 1)}

2.14 ANSWERS TO CHECK YOUR PROGRESS

1. 2^{12}

2. I_A = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}

3. {(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d) (b, c, a), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c, d, d)}

4. R^{-1} = {(7, 4), (8, 5), (7, 6), (8, 6), (9, 6)}

5. (a) None; (b) (2) and (3); (c) (1) and (4); (d) all except (3).

6. (b) {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}.

7. Equivalence relation.

8. One-one, but not onto.

9. (a) {(1, 1), (2, 4), (3, 3), (4, 3)}; (b) {(1, 1), (2, 2), (3, 1), (4, 1)}; (c) {(1, 4), (2, 3), (3, 3), (4, 4)}.

10. (a) *g*, *k*; (b) *f*, *g*, *h*; (c) *g*; (d) none.

11.Yes, since set inclusion is reflexive, antisymmetric, and transitive.

That is, for any sets *A*, *B*, *C* in _ we have:

(i) $A \subseteq A$;

(ii) if $A \subseteq B$ and $B \subseteq A$, then $A = B$;

(iii)if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
Course Code: MCA-25	**AUTHOR: RENU BANSAL**
Lesson No. 3	
LOGIC AND PROPOSITIONAL CALCULUS	

STRUCTURE

3.1 LEARNING OBJECTIVE

After going through this unit, you will be able to know:

1. What is logical expressions?
2. What is compound statement?
3. How to construct truth tables?
4. What are operators?
5. What is logical equivalence, tautology and contradiction?

3.2 INTRODUCTION

7.6.2 Introduction

Many algorithms and proofs use logical expressions such as:

"IF p THEN q" or "If p1 AND p2, THEN q1 OR q2"

Therefore it is necessary to know the cases in which these expressions are TRUE or FALSE, that is, to know the "truth value" of such expressions.

In logic we are interested in true or false of statements, and how the truth/falsehood of a statement can be determined from other statements. However, instead of dealing with individual specific statements, we are going to use symbols to represent arbitrary statements so that the results can be used in many similar but different cases. The formalization also promotes the clarity of thought and eliminates mistakes. There are various types of logic such as logic of sentences (propositional logic), logic of objects (predicate logic), logic involving uncertainties, logic dealing with fuzziness, temporal logic etc. Here we are going to be concerned with propositional logic and predicate logic, which are fundamental to all types of logic.

3.3 PROPOSITIONS AND COMPOUND STATEMENTS

Proposition

A proposition (or statement) is a declarative statement which is true or false, but not both.

Sentences considered in propositional logic are not arbitrary sentences but are the ones that are either true or false, but not both. This kind of sentences are called propositions. If a proposition is true, then we say it has a truth value is "true"; if a proposition is false, its truth value is "false".

Consider the following six sentences:

(i) Grass is green.

(ii) 2 + 2 = 6

(iii) China is in Europe.

(iv) 6 + 4 = 5

(v) What is your name?

(vi) Keep quiet.

The first four are propositions, the last two are not. Also, (i) and (iii) are true, but (ii) and (iv) are false.

Compound Propositions

In everyday life we often combine propositions to form more complex propositions or we can say compound proposition. For example combining "Grass is green", and "The sun is red" we say something like "Grass is green and the sun is red", "If the sun is red, grass is green", "The sun is red and the grass is not green" etc. Here "Grass is green", and "The sun is red" are propositions, and form them using connectives "and", "if... then ..." and "not" a little more complex propositions are formed. These new propositions can in turn be combined with other propositions to construct more complex propositions. They then can be combined to form even more complex propositions.

Many propositions are composite, that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound propositions

The following two propositions are composite:

“Roses are red and violets are blue.” and “John is smart or he studies every night.”

The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its sub propositions together with the way in which they are connected to form the compound propositions.

3.4 BASIC LOGICAL OPERATIONS

Simple sentences which are true or false are basic propositions. Larger and more complex sentences are constructed from basic propositions by combining them with connectives. Thus propositions and connectives are the basic elements of propositional logic. Though there are many connectives, we are going to use the following three basic connectives or operators here:

AND, OR, NOT

Respectively known as conjunction, disjunction, and negation.

Conjuction (AND) (∧):

Conjunction, $p \wedge q$ Any two propositions can be combined by the word "and" to form a compound proposition called the conjunction of the original propositions. Symbolically, $p \wedge q$ read "p and q," denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q as follows:

If p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false. The truth value of $p \wedge q$ may be defined equivalently by the table in Fig. 3-1(a). Observe that there are four lines corresponding to the four possible combinations of T and F for the two subpropositions p and q. Note that $p \wedge q$ is true only when both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	$\neg p$
T	F
F	T

(c) "not p"

Fig. 3-1

EXAMPLE 4.1 Consider the following statements: (i) Ice floats in water and 2 + 2 = 4. (ii) Ice floats in water and 2 + 2 = 5. Only the first statement is true and the others is false since at least one of its substatement is false.

Disjunction (or) (∨):

Any two propositions can be combined by the word "or" to form a compound proposition called the disjunction of the original propositions. Symbolically, $p \vee q$ read "p or q," denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is true. The truth value of $p \vee q$ may be defined equivalently by the table in Fig. 3-1(b). Observe that $p \vee q$ is false only in the fourth case when both p and q are false.

EXAMPLE 4.2 Consider the following statements:

(i) Ice floats in water or 2 + 2 = 4 (ii) Ice floats in water or 2 + 2 = 5.

Both the statement are true because at least one of its sub-statements is true.

Negation,(NOT), ¬p

Given any proposition p, another proposition, called the negation of p, can be formed by writing "It is not true that ..." or "It is false that ..." before p or, if possible, by inserting in p the word "not." Symbolically, the negation of p, read "not p," is denoted by

$$\neg p$$

The truth value of $\neg p$ depends on the truth value of p as follows:

If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true.

The truth value of $\neg p$ may be defined equivalently by the table in Fig. 3-1(c). Thus the truth value of the negation of p is always the opposite of the truth value of p.

EXAMPLE 4.3 Consider the following statements:

(1) sun rises in the east. (2) It is false that sun rises in the east. (3) sun does not rises in the east

Then (2) and (3) are each the negation of (1).

3.5 PROPOSITIONS AND TRUTH TABLES

Truth Tables

An expression constructed from logical variables (p, q,...,) will be called a proposition. The main property of a proposition P (p, q, . . .) is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a truth table. We describe a way to obtain such a truth table below.

Consider, for example, the proposition ¬(p ∧ ¬q). Figure 3-2(a) indicates how the truth table of ¬(p ∧ ¬q) is constructed. Observe that the first columns of the table are for the variables p, q,... and that there are enough rows in the table, to allow for all possible combinations of T and F for these variables. (For 2 variables, as above, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in general, for n variables, 2n rows are required.) There is then a column for each "elementary" stage of the construction of the proposition, the truth value at each step being determined from the previous stages by the definitions of the connectives ∧, ∨, ¬. Finally we obtain the truth value of the proposition, which appears in the last column.

The actual truth table of the proposition ¬(p∧¬q)is shown in Fig. 3-2(b). It consists precisely of the columns in Fig. 4-2(a) which appear under the variables and under the proposition; the other columns were merely used in the construction of the truth table.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(a)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(b)

Fig: 3-2

3.6 TAUTOLOGIES AND CONTRADICTIONS

Tautology and Contradiction

A **TAUTOLOGY** is a formula which is "always true" --- that is, it is true for every assignment of truth values to its simple components.

The opposite of a tautology is a **CONTRADICTION**, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components.

Some propositions P (p, q, . . .) contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies.

For example:

Proposition [$p \vee \neg(p \wedge q)$] is tautology. As if we construct the truth table of $p \vee \neg(p \wedge q)$ as shown in Fig.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since the truth value of $p \vee \neg(p \wedge q)$ is T for all values of p and q, the proposition is a tautology.

Analogously, a proposition P (p, q, . . .) is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example,

the proposition "p and not p," that is, $p \wedge \neg p$, is a contradiction. This is verified by looking at their truth tables in following Fig.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

3.7 LOGICAL EQUIVALENCE

Logical Equivalence

Two propositions P (p, q, . . .) and Q(p, q, . . .) are said to be logically equivalent, or simply equivalent or equal, denoted by

P (p, q, . . .) ≡ Q(p, q, . . .)

If they have identical truth tables. Consider, for example, the truth tables of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ appearing in Fig. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

In other words, the propositions are logically equivalent.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

Note:

logical equivalence can also apply for compound statements, for example:

Let p be “he is rich” and q be “he is happy” Let S be the statement:

“It is not true that he is rich and he is happy.”

Then S can be written in the form $\neg(p \wedge q)$. However, as noted above, $\neg(p \wedge q) \equiv \neg p \vee \neg q$. Accordingly, S has the same meaning as the statement:

“he is not rich, or he is not happy.”

3.8 ALGEBRA OF PROPOSITIONS

ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table 4-1. (In this table, T and F are restricted to the truth values "True" and "False," respectively.) We state this result formally.

Theorem 4.2: Propositions satisfy the laws of Table 4-1.

Table 4-1 Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

3.9 CONDITIONAL AND BICONDITIONAL STATEMENTS

Conditional Statement

Let p and q are two statements and these two statements are of the form "If p then q." Then Such statements are called conditional statements and are denoted by

$$p \rightarrow q$$

the conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q."

The implication $p \rightarrow q$ is false only when p is true, and q is false; otherwise, it is always true. In this implication, p is called the hypothesis (or antecedent) and q is called the conclusion (or consequent).

The truth values of $p \rightarrow q$ are defined by the tables in following Fig(a).

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \to q$

The followings are conditional statements.

1. If a = b and b = c, then a = c.
2. If I get money, then I will purchase a computer.

Bi Conditional Statement

If p and q are two statements then "p if and only if q" is a compound statement, denoted as $p \leftrightarrow q$ and referred as a biconditional statement or an equivalence. The equivalence $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false.

biconditional statements and are denoted by:

$$p \leftrightarrow q$$

For Example: (i) Two lines are parallel if and only if they have the same slope.
(ii) You will pass the exam if and only if you will work hard.

These two statements are example of biconditional statements.

The truth values of $p \leftrightarrow q$ are defined by the tables in below Fig. Observe that:

The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note :The truth table of $\neg p \wedge q$ appears in following Fig. Note that the truth table of $\neg p \vee q$ and $p \rightarrow q$ are identical, that is, they are both false only in the second case. Accordingly, $p \rightarrow q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

3.10 CHECK YOUR PROGRESS

Q1. Let *p* be “It is cold” and let *q* be “It is raining”. Give a simple verbal sentence which describes each of the following statements: (*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

In each case, translate ∍, (, and <to read “and,” “or,” and “It is false that” or “not,” respectively, and then simplify the English sentence.

Q2. Verify that the proposition *p* (¬ *p* is a tautology.

Q3 Show that the propositions ¬ *(p* ∍*q)* and ¬ *p* (¬ *q* are logically equivalent.

Q4. Rewrite the following statements without using the conditional:

(*a*) If it is cold, he wears a hat.

(*b*) If productivity increases, then wages rise.

Q5. Which of the following propositions are tautologies? Which are contradictions? Why?

(a) It is raining or it is not raining.

(b) It is raining (P) and it is not raining (¬P).

Q6. Use the laws in Table 4-1 to show that $\neg (p \ni q)\ ((\neg p \ni q) \equiv \neg p$.

3.11 Summary

1. A proposition (or statement) is a declarative statement which is true or false, but not both.

2. Many propositions are composite, that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound propositions

3. Though their are three basic connectives or operators known as AND, OR, not

respectively known as conjunction, disjunction, and negation.

4. Property of proposition is that its truth value depends upon the truth value of its variables.

5. If propositions have true for any truth value of their variables, such propositions are called tautaology.

6. If propositions have false for any truth value of their variables, such propositions are called contradiction.

7. Two propositions are said to be logically equivalent if they have identical truth tables.

8. The conditional statement are denoted by $p \rightarrow q$ and is frequently read "p implies q".

9.biconditional statements are denoted by $p \leftrightarrow q$ and is frequently read as "p if and only if q"

3.12 Key words

Propositions

Tautology

Contradiction

Conditional statements

Biconditional statements

Logical equivalence

3.13 ANSWER TO CHECK YOUR PROGRESS

Ans. 1(a) It is not cold. (c) It is cold or it is raining.

(b) It is cold and raining. (d) It is raining or it is not cold

Ans.2 As truth table shows that all output is true, so it is a tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Ans.3 As shown in figure both tables have same output so they are logically equivalent

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

Ans.4 Recall that "If *p* then *q*" is equivalent to "Not *p* or *q*;" that is, $p \to q \equiv \neg p \vee q$. Hence,

(*a*) It is not cold or he wears a hat.

(*b*) Productivity does not increase or wages rise.

Ans.5 (a)Answer: tautology

(b)Answer: contradiction

Example reasoning: All rows in the truth table evaluate to false.

Ans.6 Statement Reason

(1) ¬ (p (q) ((¬ p ϶q) ≡ (¬ p ϶ ¬ q) ((¬ p ϶q) DeMorgan's law

(2) ≡ ¬ p э(¬ q (q) Distributive law

(3) ≡ ¬ p эT Complement law

(4) ≡ ¬ p Identity law

3.14 SELF ASSESMENT TEST

Q1.Write the negation of each statement as simply as possible:

(*a*) If she works, she will earn money.

(*b*) He swims if and only if the water is warm.

(*c*) If it snows, then they do not drive the car.

Q2. Negate each of the following statements:

(a) If the teacher is absent, then some students do not complete their homework.

(b) All the students completed their homework and the teacher is present.

(c) Some of the students did not complete their homework or the teacher is absent

Q3. Find the truth tables for. (a) $p \vee \neg q$; (b) $\neg p \wedge \neg q$.

Q4. Verify that the proposition $(p \wedge q) \wedge \neg (p \vee q)$ is a contradiction.

3.15 References/ Suggested Readings

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
Course Code: MCA-25	**AUTHOR: KAPILA DEVI**
Lesson No. 4	
GROUP THEORY	
REVISED /UPDATED SLM BY RENU BANSAL	

STRUCTURE

4.1 LEARNINGOBJECTIVES

After going through this unit, you will be able to know:

1. How to describe groups
2. Description of subgroups
3. Definitions of cosets
4. Know about cyclic groups
5. Definitions normal subgroups
6. Illustrate quotient groups.

4.2 INTRODUCTION

In this unit we will introduce the notion of groups. Group is an algebraic structure, i.e, a non- empty set equipped with a binary operation satisfying certain postulates. Group theory occupies an important position in the study of abstract algebra.

4.3 GROUP

A non-empty set G, equipped with a binary operation *, is called a group if the following postulates are satisfied.

(i) Closure property

$\forall$ a,b $\in$G : $\forall$ a*b $\in$ G

It means that for a,b belongs to G such that a*b also belongs to G

(ii) Associativity

$(a * b) * c = a * (b * c), \ \forall a, b, c \in G$

(iii) Existence of Identity

There exists $e \in G$ such that $a * e = a = e * a$, $\forall a \in G$. e is called identity.

(iv) Existence of Inverse

For every $a \in G$, there exists $b \in G$ such that $a * b = e = b * a$. so b is called inverse of a.

These all properties of group is known as group axioms.

Note: 1. In order to be a group, there must be a non-empty set equipped with a binary operation satisfying the postulates mentioned above.

2. We will drop the binary operation symbol and simply write ab. It should be keep in mind that this is not our usual multiplication.

4.4 Semi group

Semi Group

A non-empty set G equipped with a binary operation is called a semigroup, if the binary operation is associative.

For example : Consider the positive integer N. then (N,+) are semi group, since addition in N are associative.

4.5 Monoid

Monoid

A non-empty set G equipped with a binary operation is called a monoid , if the binary operation is associative as well as operation also has an identity element.

For example: Consider the positive integer N. then (N,×) are monoid , since multiplication in N are associative and it has identity element as 1.

4.6 Abelian group

Abelian group

A group G is said to be abelian if the commutative law holds, i.e., if a*b=b*a for every $a,b \in G$.

An operation * on a set S is said to be commutative or satisfy the commutative law if a*b=b*a

for any elements a,b in S.

For example: The set Z of integers is an abelian group under addition. The identity element is 0 and –a is additive inverse of a in Z. and addition is associative and commutative both.

Finite and Infinite Groups

A group G is finite if the set G is finite; otherwise it is infinite group.

Order of a group

The number of elements of a finite group G is called the order of G. Symbolically it is denoted by 0(G) or |G|.

Example 1: Let G = {1} Clearly multiplication is a binary operation on G. Moreover, all the postulates for a group are satisfied.

G is a group under multiplication. Furthermore, G is an Abelian group.

Example 2: Let G = {–1, 1},

The set consisting of the square roots of unity.

*	-1	1
-1	1	-1
1	-1	1

Fig. 3.1

(i) From the table, we see that multiplication is a binary operation on G.

(ii) We know that multiplication of real numbers is associative.

(iii) 1 is the identity under multiplication.

(iv) Every element G possesses inverse in G. 1 is the inverse of1, –1 is the inverse of –1. G is a group under multiplication.

(v) Moreover, multiplication of real numbers is commutative. G is an Abelian group.

Example : Let Z be the set of integers

(i) Let a, b$\in$Z

a+b$\in$Z, $\forall$a, b $\in$Z

i.e. addition is a binary operation on Z.

(ii) We know that multiplication of integers is associative.

(iii) We have 0$\in$Z and a+0= a=0+a, $\forall$a$\in$Z

$\therefore$0 is the identity.

(iv) Forever ya$\in$Z, there exists –a$\in$Z such that a+(–a)= 0= (–a)+a.

$\therefore$Z is a group under addition.

(v) Moreover, addition of real numbers is commutative.

$\therefore$Z is an Abelian group under addition.

Note: Z is an infinite group.

Some other examples are the following.

- Q is an Abelian group under addition.
- IR is an Abelian group under addition.
- Q_0, the set of non-zero rational numbers is an Abelian group under multiplication.
- IR_0, the set of non-zero real numbers is an Abelian group under multiplication.
- C is an Abelian group under addition.
- C_0, the set of non-zero complex numbers is an Abelian group under multiplication.

Example: LetG be the set of rational numbers excluding1.We define $*$ on G as follows.

a $*$b = a+b–ab, $\forall$a, b$\in$G

(i) $*$is a binary operation on G [Ref: Example10]

(ii) $*$is associative onG[Ref: Example 10]

(iii) We have 0 $\in$G and

$a*0 = a + 0 – a0 = a$

$0 *a = 0 + a –0a = a$

0 is the identity under $*$.

(v) For every a $\in$ G, there exists $\frac{a}{a-1}\in$ G such that

$$a * \frac{a}{a-1} = 0 = \frac{a}{a-1} * a$$

$\therefore \frac{a}{a-1}$is the inverse of a under x.

$\therefore$ G is a group commutative. [Ref: Example 10]

$\therefore$ G is an Abelian group.

Theorem : Let G be a group. Then

(i) Identity element is unique.

(ii) Inverse of each element in G is unique;

(iii) $(a^{-1})^{-1}$= a, for all a $\in$G, where a^{-1}denotes the inverse ofa,

(iv) $(ab)^{-1}= b^{-1}a^{-1}$, for all a, b$\in$G

(v) ab = ac $\Rightarrow$b = c, (left cancellation law)

ba = ca $\Rightarrow$b = c, (right cancellation law) for all a, b, c$\in$G.

Proof (i) If possible, let e, f be two identities of G.

$\therefore$ef = e = fe [$\therefore$f is identity]

Again ef = f = fe [$\therefore$e is identity]

$\therefore$ e = f

Thus, identity element is unique.

(ii) Let $a \in G$ and e be the identity of G.

If possible, let b and c be two inverses of a. ab = e = ba[∵b is inverse ofa]

ac = e = ca [∵c is inverse ofa]

Now b = be [∵e is the identity}

= b(ac) [∵e = ac]

= (ba)c [by associativity]

= ec[∵e is theidentity]

= c

Thus, inverse of a is unique.

Note: In view of the above theorem, we write 'the inverse of a', not 'an inverse of a'.

(iii) We have $aa^{-1} = e = a^{-1}a$

$\therefore a^{-1}$ is the inverse of a, and vice versa, a is the inverse of a^{-1} i.e. $(a^{-1})^{-1} = a$

(iv) Let e be the identity of G and $a, b \in G$ Now $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$

$= a\,e\,a^{-1}$

$= aa^{-1}$

$= e\ (b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b$

$= b^{-1}eb$

$= b^{-1}b$

= e

$\therefore b^{-1}a^{-1}$ is the inverse of ab, i.e. $(ab)^{-1} = b^{-1}a^{-1}$

(v) We have ab =ac

$\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b=(a^{-1}a)c$, [by associativity]

$\Rightarrow eb = ec$

$\Rightarrow b = c$, [$\because$e is the identity]

Again ba = ca

$\Rightarrow (ba)a^{-1} = (ca)\, a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$ [by associativity]

$\Rightarrow be = ce$

$\Rightarrow b = c$ [$\because$e is the identity]

Example5: If, in a group G, every element is its own inverse, prove that G is Abelian.

Solution: Let G be a group. Let a, b $\in$G

$a^{-1} = a$, $b^{-1} = b$ (given)

Now $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$ G is Abelian

4.7 SUBGROUP

Let us consider the set Z of integers. We know that Z is a group under addition.

Let 2Z = {0, ± 2,±4,....................................}be the set of even integers.

Clearly 2Z is a non-empty subset of Z. Moreover, 2Z is also a group under addition. In this case, 2Z is said to be a subgroup of Z.

Definition: A non-empty subset H of a group G is a subgroup of G if H is also a group under the same binary operation of G. Symbolically, we write $H \leq G$.

A subset H of a group is a subgroup of G if:

1. The identity element $e \in H$.

2. H is closed under the operation of G, i.e. if $a,b \in H$, then $ab \in H$.

3. H is closed under inverses, that is if $a \in H$, then $a^{-1} \in H$.

Note: (1) If e is the identity of G, {e} is also a subgroup of G, called the trivial subgroup of G. All other subgroups are nontrivial.

(2)G itself is a subgroup of G, called the improper subgroup of G. All other subgroups are proper.

Example : Let us consider G = {–1, 1, –i, i},$i = \sqrt{-1}$

We know that G is a group under multiplication,

Let H = {–1, 1}.

We have $H \subseteq G$.

Also H is group under multiplication

$\therefore H \leq G$.

The set of real numbers IR is a group under addition. The set $IR^+$of positive real numbers is a group under multiplication. IR_+ is a subset of IR, but not a subgroup of IR. Binary operations are different.

Theorem : Let H be a subgroup of a group G. Then

(i) The identity element of H is the identity element of G;

(ii) The inverse of any element of H is the same as the inverse of the element of G.

Proof: Let G be a group and $H \leq G$.

(i) If possible, let e be the identity of G, and f be the identity of H.

Let $a \in H$.

$\therefore af = a = fa$ [$\because$f is the identity of H]

Again $a \in H \Rightarrow a \in G$.

$ae = a = ea$ [$\because$e is the identity of G]

$af = ae$

$\Rightarrow f = e$ [$\because$by left cancellation law]

(ii) Let e be the identity of G e is also the identity of H.

Let $a \in H$.

If possible, let b be the inverse of a in H, and c be the inverse of a in G.

$ab = e = ba$

$ac = e = ca$

$ab = ac \Rightarrow b = c$ [$\because$by left cancellation law]

Note: If H,K are two subgroups of a group G and e is the identity of G, then e is also the identity of H as well as of K. So, H and K have at least one common element, viz., the identity element. Thus, we can conclude that there cannot be two disjoint subgroups of a group.

Theorem : A non-empty subset H of a group G is a subgroup of G if and only if $a, b \in H \Rightarrow ab^{-1} \in H$.

Proof : Let $H \leq G$ and $a, b \in H$.

$b \in H \Rightarrow b^{-1} \in H$ [$\because$H is a group]

Now $a \in H$, $b^{-1} \in H$

$\therefore ab^{-1} \in H$

Conversely, let $H \subseteq G$ such that $a, b \in H \Rightarrow ab^{-1} \in H$

(i) Now $a, a \in H \Rightarrow aa^{-1} \in H$ (by the given condition)

$\Rightarrow e \in H$, where e is the identity of G.

(ii) $e \in H, a \in H \Rightarrow ea^{-1} \in H$

$\Rightarrow a^{-1} \in H$,

every element of H has inverse in H.

(iii) $b \in H \Rightarrow b^{-1} \in H$ by(ii)

Now $a \in H$, $b^{-1} \in H$

$\Rightarrow a(b^{-1})^{-1} \in H$ (Since $(b^{-1})^{-1} \in H$)

$\Rightarrow ab \in H$

(iv) The binary operation is associative in H as it is associative in G.

$\therefore$ H is a group under the same binary operation in G. Also $H \subseteq G$.

$\therefore H \leq G$.

Note: If G is an additive group, i.e., if G is a group under addition, then a non-empty subset H of G is a subgroup of G if and only if $a, b \in H \Rightarrow a - b \in H$.

Theorem : The intersection of two subgroups of a group is again a subgroup of the group.

Proof: Let G be a group and e be the identity of G.

Let $H \leq G$, $K \leq G$.

We note that $H \cap K \neq \phi$,

for at least the identity element $e \in H \cap K$ Let $a, b \in H \cap K$

$a \in H \cap K \Rightarrow a \in H$ and $a \in K$ $b \in H \cap K \Rightarrow b \in H$ and $b \in K$

Now $a, b \in H$ and $H \leq G$.

$ab^{-1} \in H$ $a, b \in K$ and $K \leq G$.

$ab^{-1} \in K$ $ab^{-1} \in H$ and $ab^{-1} \in K \Rightarrow ab^{-1} \in H \cap K$

$H \cap K \leq G$

Theorem : The union of two subgroups is a subgroup if and only if one of them is contained in the other.

Proof: Let G be a group and $H \leq G$, $K \leq G$. Let $H \subseteq K$. Then $H \cup K = K$.

$\because K \le G, H \cup K \le G$

Conversely, let $H \le G$, $K \le G$ such that $H \cup K \le G$. Let e be the identity of G.

We are to prove that either $H \subseteq K$ or $K \subseteq H$

If possible, let it not be true.

$H \not\subseteq K$ and $K \not\subseteq H$

there exists $a \in H$ such that $a \notin K$ Also, there exists $b \in K$ such that $b \notin H$

Now $a \in H \Rightarrow a \in H \cup K$

$b \in K \Rightarrow b \in H \cup K$ $H \cup K \le G$,

$ab \in H \cup K$

This $\Rightarrow ab \in H$ or $ab \in K$

If $ab \in H$, then $a \in H, ab \in H$ $\Rightarrow a^{-1}(ab) \in H$

$\Rightarrow (a^{-1}a)b \in H$

$\Rightarrow eb \in H$

$\Rightarrow b \in H$,

which is a contradiction.

Similarly, if $ab \in K$, then $ab \in K$, $b \in K \Rightarrow (ab)b^{-1} \in K$

$\Rightarrow a(bb^{-1}) \in K$

$\Rightarrow ae \in K$

$\Rightarrow a \in K$

which is acontradiction.

So, our assumption is wrong. either $H \subseteq K$ or $K \subseteq H$.

4.8 COSET

Let G be a group and $H \leq G$

Let $a \in G$

The set $Ha = \{ha: h \in H\}$ is called a right Coset of H in G.

Similarly, the set $aH = \{ah: h \in H\}$ is called a left Coset of H in G.

Note:(1) Right (or left) Coset cannot be empty we have $e \in H$ $ea = a \in Ha$ $Ha \neq \phi$

Similarly, $ae = a \in aH$ $aH \neq \phi$

(2) $H \leq G$, but H a or aH is not necessarily a subgroup of G.

(3) If H is a subgroup of the additive group G and $a \in G$, the right Coset of H in G is given by $H+a=\{h+a:h \in H\}$

Similarly, the left Coset of H in G is given by $a+H=\{a+h:h \in H\}$

Example: Let us consider the additive group of integers Z. We know that $2Z = \{0, \pm 2, \pm 4, \pm 6, \ldots\} \leq Z$

We have

$0+2Z = \{0, \pm 2, \pm 4, \pm 6, \ldots\ldots\ldots\} = 2Z$

$1+2Z = \{\pm 1, \pm 3, \pm 5, \pm 7, \ldots\ldots\ldots\}$

$2+2Z = \{0, \pm 2, \pm 4, \pm 6, \ldots\ldots\ldots\}$

$3+2Z = \{\pm 1, \pm 3, \pm 5, \pm 7, \ldots\ldots\ldots\}$ and soon.

Thus, we see that

$2Z = 2+2Z = 4+2Z = \ldots$

$1+2Z = 3+2Z = 5+2Z = \ldots$

2Z has two distinct left cosets viz. 2Z and 1+2Z suchthat $2Z \cup (1+2Z) = Z$

$2Z \cap (1+2Z) = \phi$

Similarly,2Z has two distinct right cosets 2Z and 2Z+1 such that $2Z \cup (2Z+1) = Z$

$2Z \cap (2Z+1) = \phi$

Theorem : Let G be a group and $H \leq G$. Then

(i) $aH = H \Leftrightarrow a \in H$; $Ha = H \Leftrightarrow a \in H$

(ii) $aH = bH \Leftrightarrow a^{-1}b \in H$; $Ha = Hb \Leftrightarrow ab^{-1} \in H$, where $a, b \in G$

Proof: (i) Let aH = H

We have $ae \in aH$, e is the identity of G

$\Rightarrow ae \in H$ [aH =H]

$\Rightarrow a \in H$

Conversely, let $a \in H$ Let $x \in aH$

This $\Rightarrow x = ah$, for some $h \in H$

Now $a \in H, h \in H$ $ah \in H \Rightarrow x \in H$ Thus, $x \in aH \Rightarrow x \in H$

$aH \subseteq H$ ...(1)

Again, let $y \in H$ $a \in H$, $a^{-1} \in H$ $a^{-1}y \in H$

$\Rightarrow a^{-1}y = h_1$, for some $h_1 \in H$

$\Rightarrow a(a^{-1}y) = ah_1$

$\Rightarrow (aa^{-1})y = ah_1$

$\Rightarrow ey = ah_1$

$\Rightarrow y = ah_1 \in aH$

Thus, $y \in H \Rightarrow y \in aH$

$H \subseteq aH$ ...(2)

From (1) and (2), $aH = H$ Similarly $Ha = H \Leftrightarrow a \in H$.

(ii) $aH = bH$

$\Leftrightarrow a^{-1}(aH) = a^{-1}(bH)$

$\Leftrightarrow (a^{-1}a)H = (a^{-1}b)H$

$\Leftrightarrow eH = (a^{-1}b)H$

$\Leftrightarrow H = (a^{-1}b)H$

$\Leftrightarrow a^{-1}b \in H$ (using (1) Similarly, $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

Theorem : Any two left (or right) cosets of a subgroup are either disjoint or identical.

Proof: Let G be a group and $H \leq G$. Let $a, b \in G$.

Then aH, bH are two left cosets of H in G. Clearly aH, bH are either disjoint or not disjoint. Let aH, bH be not disjoint, i.e, $aH \cap bH \neq \phi$.

Let $x \in aH \cap bH$

$\Rightarrow x \in aH$ and $x \in bH$

Now $x \in aH \Rightarrow x = ah_1$ for some $h_1 \in H$

$x \in bH \Rightarrow x = bh_2$ for some $h_2 \in H$ $ah_1 = bh_2$

$\Rightarrow (ah_1)h_1^{-1} = (bh_2)h_1^{-1}$

$\Rightarrow a\, h_1h_1^{-1} = b\, h_2h_1^{-1}$

$\Rightarrow ae = bh_2h_1^{-1}$, where e is the identity of G

$\Rightarrow a = bh_2h_1^{-1}$ Similarly $b = ah_1h_2^{-1}$

Let p $\in aH \Rightarrow$ p = ah_3 for some $h_3 \in H$

$= bh_2h_1^{-1}h_3$

$\in bH, \quad h_2h_1^{-1}h_3 \in H$

Thus, p $\in aH \Rightarrow p \in bH \quad aH \subseteq bH$

(1) Again, let q $\in bH \Rightarrow q = bh_4$ for some $h_4 \in H$

$\Rightarrow q = ah_1h_2^{-1}h_4$

$\in aH, \quad h_1h_2^{-1}h_4 \in H$

Thus $q \in bH \quad \Rightarrow q \in aH \quad bH \subseteq aH$

(2) From (1) and (2), aH =bH

Thus, aH, bH are either disjoint or identical.

4.9 LAGRANGE's THEOREM

Lagrange's subgroup Order Theorem

Theorem: The order of a subgroup of a finite group divides the order of the group.

Proof : Let G be a finite group, and let O(G) = n and $H \leq G$.

Let e be the identity of G.

Case 1 : Let H = {e},

There is nothing to prove. O(H) =1, and 1 divides n.

Case 2 : Let H = G. Hence O(H) = O(G). There is nothing to prove.

Case 3 : Let H $\neq$ {e}, H $\neq$ G. Let O(H) = m, m < n.

Let H = $\{h_1, h_2, h_3, \ldots h_m\}$, where the h_i's are distinct. Let a $\in G$, a $\notin H$.

Let us consider the left Coset. aH = $\{ah_1, ah_2, ah_3, \ldots, ah_m\}$

We claim that all the elements in aH are distinct, for if $ah_i = ah_j$, $1 \le i \le m$, $1 \le$

$j \le m$, $i \ne j$ then $h_i = h_j$, (by left cancellation law), which is a contradiction. Again, no element of H is equal to any element of aH, for if $h_i = ah_j$,

then $h_i h_j^{-1}$ i.e. $h_i h_j^{-1} = a$ i.e. $a = h_i h_j^{-1} \in H$

i.e. $a \in H$, which is a contradiction.

Thus, we have listed m+m = 2m elements of G. If this exhausts all the elements of G, then 2m=n.m dividesn.

If not, let $b \in G$, $b \notin H$, $b \notin aH$

Let us consider $bH = \{bh_1, bh_2, ..., bh_m\}$

Clearly, all the elements in bH are distinct.No element of H is equal to any element of bH.

Again, no element of aH is equal to any element of bH,

For if $ah_i = bh_j$, then $ah_i h_j^{-1} = bh_j h_j^{-1}$ i.e $ah_i h_j^{-1} = b$ i.e $b = ah_i h_j^{-1} \in aH$,

$(h_i h_j^{-1} \in H)$ i.e. $b \in aH$, which is acontradiction.

Thus, we have listed m+m+m = 3m elements of G. If this exhausts all the elements of G, then 3m=n. m divides n.

If not, we proceed as above. Since G is finite, we must stop somewhere, say, after k times.

$mk = n \Rightarrow m$ divides n.

Remark: The converse of Lagrange's theorem is not true.

4.10 NORMALSUBGROUP

Definition: A subgroup H of a group G is said to be a normal subgroup of G if aH=Ha, for all $a \in G$, i.e., if its left and right cosets coincide. Symbolically we write $H \Delta G$.

Note: A group G has at least two normal subgroups, viz., G and {e}. A group, having no normal subgroups other than G and {e},is called a simple group.

Example: Let G ={ –1, 1, –i, i}

We know that G is a group under multiplication. Let H = {–1, 1}, and $H \leq G$

Now 1H = {–1, 1} = H1

(–1)H = {1, –1} = H (–1)

iH = {–i, i} = Hi

(–i)H = {i, –i} = H (–i)

Thus, aH = Ha, for all $a \in G$ $H \Delta G$.

Note: In the above example, H has two distinct left (or right) cosets, viz H, iH. We note that iH is not a subgroup of G.

Theorem : Every subgroup of an Abelian group is normal.

Proof: Let G be an Abelian group. Let $H \leq G$.

Let $a \in G$

Now aH = {ah : $h \in H$}

= {ha :$h \in H$} [G is aAbelian]

= Ha$H \Delta G$.

Theorem : The intersection of any two normal subgroup of a group is again a normal subgroup of the group.

Proof: Let G be a group, an $H \Delta G, K \Delta G$.

$H \leq G$, $K \leq G$ and so $H \cap K \leq G$.

Let $g \in G$, $h \in H \cap K$

$h \in H \cap K \Rightarrow h \in H$ and $h \in K$ Now $g \in G, h \in H$ and $H \Delta G, ghg^{-1} \in H$

Again $g \in G$, $h \in K$ and $K \Delta G$ $ghg^{-1} \in K$

Thus, $ghg^{-1} \in H$ and $ghg^{-1} \in K$. $ghg^{-1} \in H \cap K$

$H \cap K \Delta G$.

Note: The union of two normal subgroups is not necessarily a normal subgroup.

4.11 CYCLICGROUPS

Definition: A group G is said to be cyclic if there exist an element a in G such that every element of G can be expressed in the form $a^n, n \in Z$. a is said to be a generator of G. Symbolically we write G=<a>,i.e.,G is a cyclic group generated by a.

Note: If the group G is additive, every element of G is expressed in the form na.

Examples

1. We know that G={–1,1}is a group under multiplication. We have $(-1)^1=-1$, $(-1)^2=1$

   G is a cyclic group generated by –1, i.e. G = <–1>

2. We know that G={1, , ω^2} is a group under multiplication; is a complex cube root of unity. We have, $\omega^1 = \omega$, $\omega^2 = \omega^2$, $\omega^3 = 1$.. G is cyclic group generated by i.e., G=< > Similarly G = $<\omega^2>$

3. We know that G={–1,1,–i,i}is a group under multiplication. We have $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ G is a cyclic group generated by i , i.e. G = <i> Similarly G = < – i>

4. Let Z be the additive group of integers.

   Let $x \in Z$ Then x =1x

   G is a cyclic group generated by1,i.e. G = < 1 >Similarly G = < –1>.

Theorem : Every cyclic group is Abelian.

Proof : Let G = < a > be a cyclic group.

Let $x, y \in G$.

Then there exist integers m, n such that $x = a^m$, $y = a^n$Now $xy = a^m a^n$

$=a^{m+n}$

$=a^{n+m}$

$= a^n a^m$

$= yx$ therefore G is Abelian.

Theorem : If a is a generator of a cyclic group G, $a^{-1}$is also a generator of G.

Proof : Let G = < a > be a cyclic group. Let x, y ∈G.

Then there exists integer m such that $x = a^m$

$= (a^{-1})^{-m}$

$G = < a^{-1} >$.

4.12CHECK YOUR PROGRESS

1. Is ∗ defined by $a * b = \frac{a}{b}$ binary operation on IR?
2. Is multiplication a binary operation on{–1,1,i,–i}?where $i = \sqrt{-1}$
3. Is the multiplicative group IR_+ of positive real numbers a subgroup pf the additive group IR of real numbers? Justify your answer.
4. Can there be two disjoint subgroups of a group? Justify your answer.
5. Can an Abelian group have a non-Abelian subgroup?
6. Can a non-Abelian group be cyclic?
7. Let G = {1, ω ,ω^2 } be the multiplicative group; being a complex cube root of unity. Is the group simple?

4.13 SUMMARY

1. A binary operation $*$ on a non-empty set S is a function from SxS to S.
2. If S is a finite set having m elements, the number of binary operations on S is m^{m^2}.
3. A binary operation $*$on a non-empty set S is commutative if $a * b = b * a, \forall a,b \in S$.
4. A binary operation $*$on a non-empty set S is associative
5. if $(a * b)*c = a *(b * c), \forall a, b, c \in S$
6. An element e in a non-empty set S equipped with a binary operation $*$ is called an identity element for x if $a * e = a = e * a, \forall a \in S$.
7. An element b in a non-empty set S equipped with a binary operation $*$ is called an inverse element of $a \in S$ if $a * b = e = b * a$.
8. A group is an algebraic structure i.e., a non-empty set equipped with a binary operation satisfying certain postulates.
9. The order of a group G is the number of elements in G and the order of an element in a group is the least positive integer n such that an is the identity element of that group G.
10. A non-empty subset H of a group G is a subgroup of G if H is also a group under the same binary operation of G.
11. If H is a subgroup of a group G, the set $Ha = \{ha: h \in H\}$, $a \in G$, is called a right coset of H in G. Similarly,a left coset is defined.
12. A group G is said to be cyclic if there exists an element a in G such that every element of G can be expressed in the form $a^n; n \in Z$
13. A subgroup H of a group G is a normal subgroup of G if aH=Ha, for all $a \in G$.

4.14KEYWORDS

Semigroup: A finite or infinite set 'S′ with a binary operation '*' (Composition) is called semigroup if it holds following two conditions simultaneously −

- Closure − For every pair (a, b) ∈S, (a*b) has to be present in the set S.
- Associative –For every element a, b, c ∈S, (a*b) *c=a*(b*c) must hold.

Monoid: A monoid is a semigroup with an identity element. The identity element (denoted by e or E) of a set S is an element such that (a*e) = a, for every element a∈S. An identity element is also called a unit element. So, a monoid holds three properties simultaneously − Closure, Associative, Identity element.

Group: A group is a monoid with an inverse element. The inverse element (denoted by I) of a set S is an element such that (a*I) =(I*a) =a, for each element a∈S. So, a group holds four properties simultaneously - Closure, Associative, Identity element, Inverse element.

Abelian group: An abelian group G is a group for which the element pair (a,b)∈G always holds commutative law. So, a group holds five properties simultaneously - Closure, Associative, Identity element, Inverse element, Commutative.

Cyclic group: A group that can be generated by a single element. Every element of a cyclic group is a power of some specific element which is called a generator. A cyclic group can be generated by a generator 'g', such that every other element of the group can be written as a power of the generator 'g'.

Subgroup: A subgroup H is a subset of a group G (denoted by H≤G) if it satisfies the four properties simultaneously − Closure, Associative, Identity element, and Inverse.

4.15 ANSWERS TO CHECK YOUR PROGRESS

1. No.
2. Yes
3. No,Binary operations are different.
4. No. Two subgroups of a group have at least one common element, viz., the identity element.
5. No.
6. Every cyclic group is Abelian. Therefore, every non-Abelian group is non-cyclic.
7. Yes. It has no normal subgroups other than G, {1}.

4.16 SELF ASSESSMENT TEST

1. Let IR be the set of real numbers. Examine if the binary operation $*$ defined by $a*b=a+5b$; $a,b\in IR$ is (i)commutative, (ii)associative.
2. Let S be a set having 3 elements.How many binary operations can be defined on S?
3. Let G be the set of odd integers. A binary operation $*$on G is defined as follows $a*b = a+b-1$; a, $b \in G$.Show that G is an Abelian group under $*$.
4. Prove that a group G is Abelian if and only if $(ab)^{-1}=a^{-1}b^{-1}$,for all $a,b\in G$.
5. Let G be an Abelian group.Prove that the set $H=\{x\in G:x=x^{-1}\}$is a sub group of G.
6. Show that a group of order 4 is always Abelian.
7. Let g be a group and e be the identity of G.Prove that G is Abelian if $b^{-1}a^{-1}b\,a = e$, for all a, $b \in G$.
8. Let G be a group and H be a subgroup of G. Let $a,b\in G$. Show that the two right cosets Ha,Hb are equal if and only if the two left cosets $a^{-1}H$, $b^{-1}H$ are equal.

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
Course Code: MCA-25	**AUTHOR: PANKAJ KUMAR**
Lesson No. 5	
GRAPH THEORY	
REVISED /UPDATED SLM BY RENU BANSAL	

STRUCTURE

5.1 LEARNINGOBJECTIVES

After going through this unit you will be able to know

1. To gain some knowledge about graphs.
2. Types of graph
3. Representation of graph
4. Graph traversal algorithms
5. What is Planar Graph

5.2 INTRODUCTION

In this chapter we have defined graph which is pictorial representation of relations on sets. We have defined directed graph, undirected graphs, paths, circuits and matrix associated with graphs.

5.3 GRAPH

A pair of set {V, E}, $V \neq \phi$, constitute a graph. Elements of set V are called vertices while elements of set E are called edges or lines or curves. Generally, lines and points of plane represent the edges and vertices of the graph.

Note: 1. If V is a finite set then we say that graph is finite graph.

2. Each edge is represented by a pair of vertices say u and v, these vertices are called end points of edges we will denote it by $E_{(u, v)}$.

5.3.1 DIRECTED GRAPHS

If we put u and v as an ordered pair, then edge is called directed from u to v and such a graph in which each edge is directed is called directed graph. For example, figures given below is directed graph.

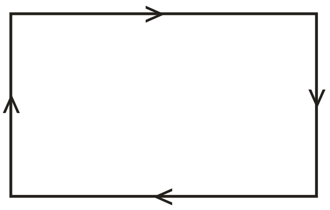


FIG 5.1 DIRECTED GRAPH

5.3.2 UNDIRECTED GRAPH

If a graph is not directed is called undirected graph in such graph edges are given as $E_{(u, v)}$. Graphs given in Figure 5.2are undirected graphs.

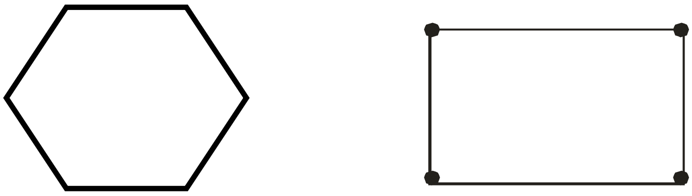
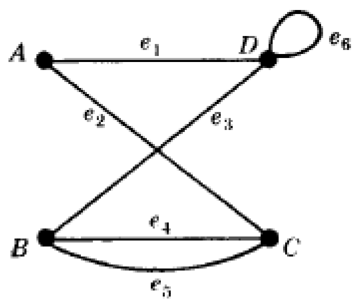


FIG 5.2 UNDIRECTED GRAPHS

5.3.3. MULTI GRAPH

Consider the diagram, The edges e4 and e5 are called multiple edges since they connect the

same endpoints, and the edge e6 is called a loop since its endpoints are the same vertex. Such a diagram is called a multigraph.



Note: (1) Edge E $_{(u,\ u)}$ is called a self-loop. A graph with no self loop and no parallel edge is called a simple graph otherwise it is called non-simple or multi graph. For example, graph of a relation which is neither reflexive nor symmetric is simple and that of reflexive relation is not simple. Figure (a) and (b) are graph which are non-simple and simple respectively.

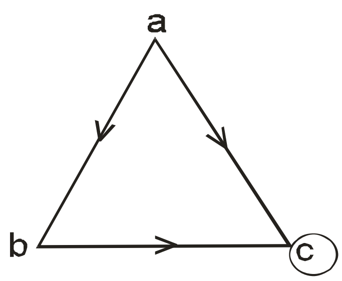


FIG (A)

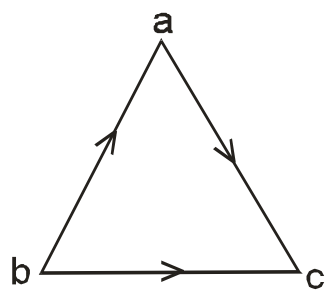


FIG (B)

(2) Edges e_1 and e_2 are parallel edges if they have same vertices. Here e_1 and e_2 are parallel edges.

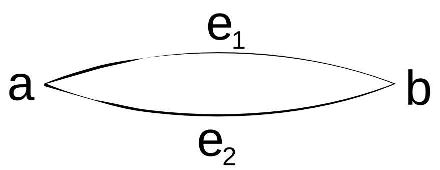


FIG 5.3 PARALLEL EDGES

5.4 PATH AND CIRCUIT

In the following figure, we have $e_{(a_1,a_2)}$, $e_{(a_2,a_3)}$, $e_{(a_3,a_4)}$ $e_{(a_4,a_5)}$, $e_{(a_5,a_6)}$ are the edges so that we move from a_1 to a_6 along these edges without using an edge more than once.

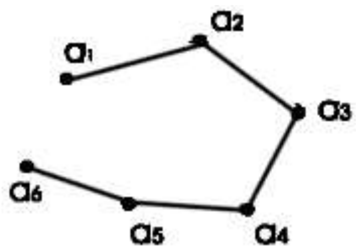


FIG 5.4 SIMPLE GRAPH

WALK: -Let G be a graph. Then a sequence of vertices v_0, v_1, v_2, ……. v_t each adjacent to the next and there is always an edge between v_i, and v_{i+1}, is called a walk. The vertex v_0 is called the initial vertex and the vertex v_t is called terminate vertex of the path. Number of edges in a walk I s called its length. A walk is called open walk if it has different beginning and end points and is called closed walk if it's beginning and end points are same.

Definition: A **Trail** is a walk having all distinct edges. A **Path** is a walk in which all vertices are distinct. A closed trail is called a **Circuit**. A circuit in which vertices (except the first and last) do not repeat is called a **Cycle.**

Note: A path is always a trail but a trail need not be a path. Similarly, a cycle is always a circuit but a circuit is not a cycle always.

In figure given below for the graph aba, one is circuit while other is not a circuit.

FIG 5.5 NOT A CIRCUIT

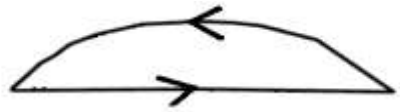


FIG 5.6 CIRCUIT AND CYCLE

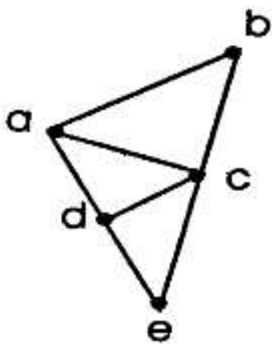


FIG 5.7

1. a b c d a c is a trail as no edge repeats
2. a b c d e , a d c b and a d e c b are paths
3. a c d e c b a is a circuit but not a cycle
4. a b c a , a b c d a and a b c e d a are cycles.

6.5 SOME DEFINITIONS

Degree of a vertex: In a non-directed graph G, the degree of a vertex v is determined by counting each loop on v twice and each other edge once. It is denoted by d (v).

Example: d (V_1) = 1, d (V_2) = 2, d (V_3) = 3 are the degrees of V_1, V_2 and V_3 in following figure.

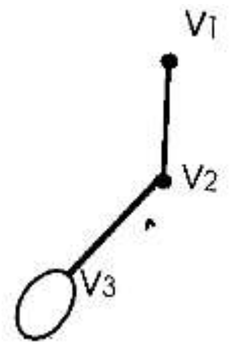


FIG 5.8

Theorem: The sum of d(v_i) for each v_i of a undirected graph G (V, E) is twice the number of edges in G

Proof: Since G is undirected graph, each edge of G is incident with two vertices, therefore, contributes 2 to the sum of degree of all the vertices of the undirected graph. Therefore, the sum of degrees of all the vertices in G is twice the number of edges in G.

Example: Draw a simple graph with three vertices i.e. draw a graph with no self-loop and no parallel edges.

Solution: Figure shown below is a simple graph.

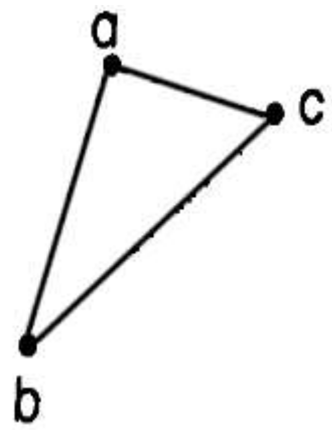


FIG 5.9

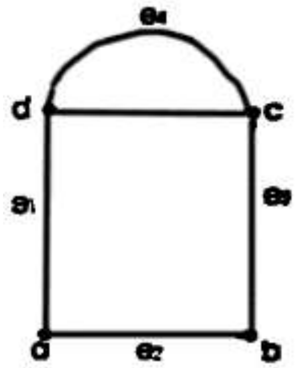


FIG 5.10

There are five edges in this graph in which edge e_4 and e_5 has same vertices (c, d). Therefore, these edges are parallel edges. Further sum of degrees of the vertices is 2+2+3+3=10 = 2.5= 2 times the number of edges.

Definitions

1. Isolated vertex: A vertex on which no edge incident is called isolated vertex, i.e. a vertex v such that d (v) = 0.

2. Null graph: A graph G = (V, E) such V $\neq\phi$ and E = ϕ is called null graph, therefore null graph in which every vertex is isolated.

Theorem 1: In a non-directed graph, the number of vertices of odd degree is always even.

Proof: Let the number of vertices in a graph G be n. Let's suppose that the degree of first k vertices say $v_0, v_1, v_2, \ldots. v_k$ be even and remaining n-k vertices be odd i.e. the vertices with odd degree.

Now $$\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) \quad \ldots (1)$$

But we know by Theorem (3.4.2) that L.H.S. of (1) is even. As $d(v_i)$ in first term of R.H.S. is even, therefore, $\sum_{i=1}^{k} d(v_i)$ is also even. It gives us that $\sum_{i=k+1}^{n} d(v_i)$ must be even. But each d (v_i) in that $\sum_{i=k+1}^{n} d(v_i)$ is odd. Moreover, we know that sum of odd number is even if they are taken even number of times. So here n-k must be even. i.e. odd number vertices in the graph must be even. Hence the proof is over.

5.6 CONNECTED AND DISCONNECTED GRAPHS.

Definition: If in a graph we can move from any vertex to the any other vertex of the graph then such graphs are called connected graphs otherwise it is called disconnected graph. Simple we can say that if there exists a path between every pair of vertices the graph is called connected.

For example, Graph in Fig. 5.13 is connected graph while in Fig. 5.14 it is disconnected.

FIG 5.11

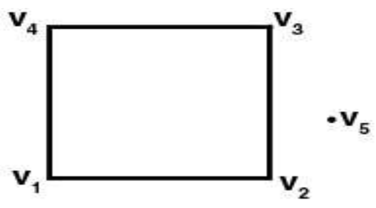


FIG 5.12

Definition:

1. Let G be a connected graph. The edge connectivity of G is the minimum number of lines (Edges) whose removal results in a disconnected or trivial graph. It is generally denoted by δ(G).

2. Vertex connectivity of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph is called the vertex connectivity of G. It is generally denoted by k(G).

Theorem. The edge connectivity of a connected graph G cannot exceed the minimum degree of G, i.e. $\lambda(G) \leq \delta(G)$.

Proof: Let G be a connected graph and v be a vertex of minimum degree in G. Then the removal of edges incident with the vertex v disconnects the vertex v from the graph G. Thus the set of all edges incident with the vertex v forms a cut set of G. But from the definition, edge connectivity is the edge connectivity of G cannot exceed the minimum degree of v, i.e. λ (G) $\leq\delta$ (G).

Theorem: The vertex connectivity of a graph G is always less than or equal to the edge connectivity of G, i.e., k (G) $\leq\lambda$ (G).

Proof: If graph G is disconnected or trivial the k (G) = λ (G) = 0. If G is connected and has a bridge e, then λ = 1. In this case K = 1, since either G has a cut point incident with e or G is K_2.($\therefore$ k (G) $\leq\lambda$ (G) when λ (G) = 0 or 1). Finally let us suppose that λ (G) $\geq$ 2. The G has λ lines whose removal disconnects G. Clearly the λ-1 of these edges produces a graph with a bridge e = {u, v}. For each of these λ-1 edges select an incident point which is different from u or v. The removal of these points (vertices) also removes λ-1 edges and if the resulting graph is disconnected then k $\leq\lambda$-1 <λ. If not the edge e = {u, v} is a bridge and hence the removal of u and v will result in either a disconnected or a trivial graph. Hence k $\leq\lambda$ in each case and this completes the proof of the theorem.

Thus, the vertex connectivity of a graph does not exceed the edge connectivity and edge connectivity of a graph cannot exceed the minimum degree of G. Hence the theorem given below.

Corollary: For any graph G, k(G) $\leq\lambda$ (G) $\leq\delta$ (G) is disconnected.

Theorem: A graph is disconnected if V can be written as union of two non-empty, disjoint subsets V_1 and V_2 such that there exist no element of E whose one vertex in V_1 and other in V_2.

Proof: Let us suppose that G be a connected graph. Take any vertex v in G. Let V_1 be the collection of all these vertices, which are joined by paths to v. Since G is not connected $V\neq V_1$ [if V= V_1 then G will be connected]. So take v_2 a set having all vertices of G which are not in all vertices of V which are not in V_1. Therefore, V_1 and V_2 are required subsets of V.

Conversely, suppose that $V = V_1 \cup V_2$, $v_1 \neq \Phi$, $v_2 \neq \Phi$, $V_1 \cap V_2 = \phi$, then if we take $v_1 \in V_1$ and $v_2 \in V_2$ then there exist no edge between v_1 and v_2 i.e. graph is disconnected.

Component of a graph means maximal connected subgraph of graph G (V, E). For example, in Fig. 5.15.

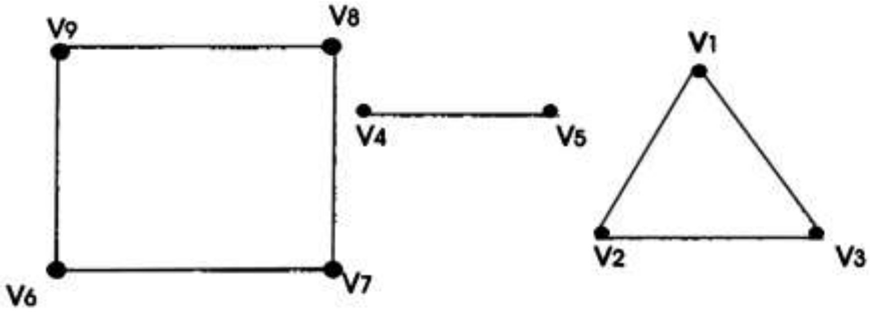


FIG 5.13

$\{v_1,v_2,v_3\}$, $\{v_4,v_5\}$, $\{v_6,v_7,v_8,v_9\}$, are components. **Further it is clear that a graph is connected if and only if it has exactly one component.**

Theorem: A simple graph with m vertices and r components can have at most (m-r) (m – r + 1)/2 edges.

Proof: Let G (V, E) be a graph with m vertices and r components let $m_0, m_1, \ldots m_r$ be the number of vertices in each components of G (V, E). Then we have

$\Sigma mi = m$ and $m_i \geq 1$ (1)

Now from (1) we get

$$\sum_{i=1}^{r}(m_i - 1) = m - r \quad (2)$$

Squaring (2) on both sides we get

$$\left(\sum_{i=1}^{r}(m_i - 1)\right)^2 = m^2 + r^2 - 2mr$$

But $\left(\sum_{i=1}^{r}(m_i - 1)\right)^2 = \sum_{i=1}^{r}(m_i - 1)^2 + 2(m_i - 1)(m_j - 1) = m^2 + r^2 - 2mr$

$\Rightarrow \sum_{i=1}^{r}(m_i - 2m_i) + r \le m^2 + r^2 - 2mr \ (\because (m_i - 1) \ge 0 \quad and \ (m_j - 1) \ge 0)$

$\Rightarrow \sum_{i=1}^{r} m_i^2 - 2\sum_{i=1}^{r} m_i \le m^2 + r^2 - 2mr - r$

$\Rightarrow \sum_{i=1}^{r} m_i^2 \le m^2 + r^2 - 2mr - r + 2m$

We also know that in a simple graph with m_i vertices have at most $m_i(m_i - 1)/2$. Thus the maximum number of edges in G is

$$\sum_{i=1}^{r}\frac{1}{2}m_i(m_i - 1) = \left[\frac{1}{2}\sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i\right] = \frac{1}{2}\left[m^2 - (r-1)(2m - r) - m\right]$$

$$< \frac{1}{2}(m - r)(m - r + 1)$$

This completes the proof.

Definition: Two vertices u and v in a digraph are said to be mutually reachable if G contains both directed u-v walk and a directed v-u walk. A digraph is said to be strongly connected if every two of its vertices are mutually reachable.

5.7MATRIX REPRESENTATION OF GRAPHS

Since we know that it is very easy to manipulate matrices. We take the matrix associated with different graphs. There are two ways of representing graph- (1) incidence matrix; (2) Adjacency matrix.

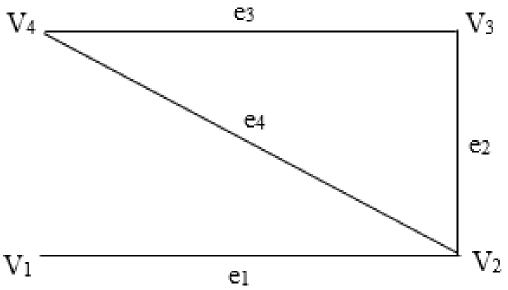
5.7.1 INCIDENCE MATRIX

Let $v_1, v_2, \ldots. v_n$ be n vertices and $e_1, e_2, \ldots, e_m$ be m edges of graph G. Then an n × m matrix I = $[a_{ij}]$ whose n rows correspond to n vertices and m columns corresponds to m edges where $a_{ij}$is as

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not an end point of edge } e_j \\ 1 & \text{if } v_i \text{ is not an end point of edge } e_j \\ 2 & \text{if } e_j \text{ is self loop on } v_i \end{cases}$$

this matrix I = $[a_{ij}]$ is called incidence matrix.

Example: Write the incidence matrix of the graph given in Figure



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

FIG 5.14 GRAPH WITH INCIDENCE MATRIX

Some facts about incidence matrix of a graph without self-loop.

1. Number of one's in each column is exactly two since each edge incident exactly on two vertices.
2. With given incidence matrix there always exists a graph.

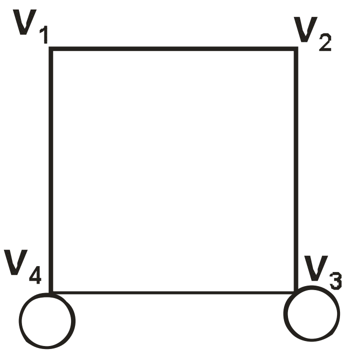
3. Sum of entries of any row of matrix gives us the degree of corresponding vertex.
4. A row with all zeroes represents an isolated vertex.

5.7.2 ADJACENCY MATRIX

Let us consider a graph with n vertices say $v_1, v_2, \ldots, v_n$. Then a square matrix $X = [x_{ij}]$ of order n, where

$$x_{ij} = \begin{cases} \text{the number of edges between } v_i \text{ and } v_j \text{ if } v_i \neq v_j \\ \text{the number of self loops at } v_i \text{ if } v_i = v_j \end{cases}$$

Example: Write the adjacency matrix of the graph given below:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

FIG 5.15 GRAPH AND ADJACENCY MATRIX

Some facts about adjacency matrix

(1) X (G) is symmetric matrix.

(2) If G has no self-loops then diagonal entries of adjacency matrix are zero. If ithdiagonal entry is 1, then it indicates that there is a self-loop at i^{th} vertex v_i.

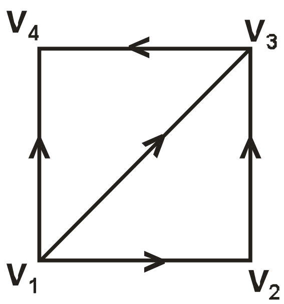
5.7.3 ADJACENCY MATRIX OF A DIGRAPH

(i.e. a directed graph) is defined as A (G) = $[x_{ij}]_{nxn}$ where G is a graph with n vertices and no parallel edges and

$$x_{ij} = \begin{cases} \text{the number of edges directed from } v_i \text{ to } v_j \text{ if } v_i \neq v_j \\ \text{the number of self loops at } v_i \text{ if } v_i = v_j \end{cases}$$

By definition it is clear that the sum of elements of i^{th} row of adjacency matrix is equal to the outgoing degree of vertex v_i i.e. the number of edges going out of vertex v_i.

Example: Write the Adjacency matrix of the diagraph given below:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

FIG 5.16 ADJACENCY MATRIX OF A DIGRAPH

5.8 SOME SPECIAL GRAPHS

5.8.1 REGULAR GRAPH

A graph in which every vertex has the same degree is called a regular graph. If every vertex has degree K, then the graph is called a k-regular or regular graph of degree K.

Note:

1. A graph is called a null graph is every vertex in the graph is an isolated vertex i.e every null graph is

regular of degree zero.

2. A complete graph k_n is regular of degree n–1.
3. If a graph has n vertices and is regular of degree k, then it has k n/2 edges.

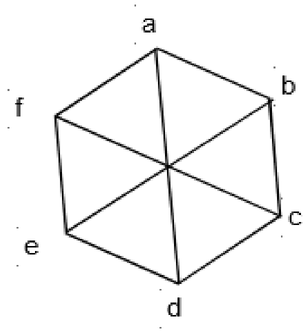
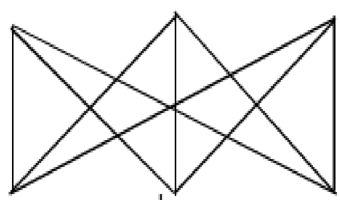


FIG 5.17 REGULAR GRAPH

5.8.2 BIPARTITE GRAPH

A simple graph is called bipartite if its vertex set V can be Partitioned into two disjoint sets $v_1$and$v_2$such that every edge in the graph connects a vertex in v_1 and a vertex in v_2. In other words, a graph G is called bipartite graph when $V=v_1 U v_2$ and $v_1 \cap v_2=\phi$ and every edge of G is of the form (v_i, v_j) with $v_i \in v_1$ and $v_j \in v_2$

A complete bipartite graph is a bipartite graph in which every vertex of v_1 is adjacent to every vertex of v_2. If number of vertices in v_1 are m and number of vertices in v_2 are n, then the complete bipartite graph is denoted by Km,n. A complete bipartite graph km,n has m+n vertices and mn edges. The complete bipartite graph $k_{3.3}$ and $k_{3,4}$ are displayed in fig.6.20



$k_{3,3}$

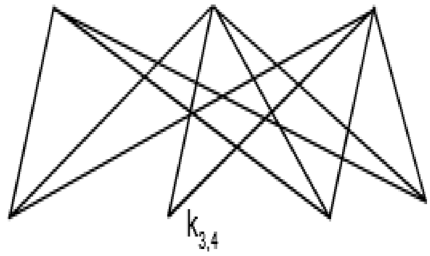


FIG 5.18 BIPARTITE GRAPHS

5.8.3 CONNECTED GRAPHS

A graph is said to be connected if we can reach any vertex from any other vertex by traveling along the edges. More formally, A graph is said to be connected if there exists at least one path between every pair of its vertices, otherwise, the graph is disconnected. That is a graph G is connected if give any vertices u and v, it is possible to travel from u to v along a sequence of adjacent edges fig.6.21 is connected graph but in fig.6.22 is a disconnected graph.

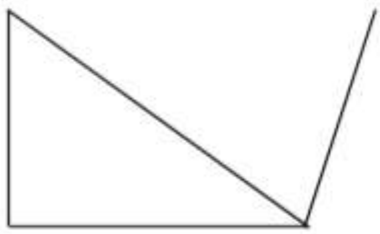


FIG 5.19 CONNECTED GRAPH

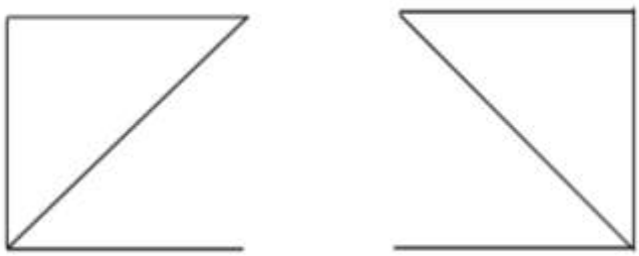


FIG 5.20 DISCONNECTED GRAPH

A disconnected graph consists of two or more connected graphs. Each of these connected subgroups is called a component. Figure 5.22 is a disconnected graph with two components.

5.8.4 EULER GRAPH

An undirected graph with no isolated vertices is said to have an Euler circuit if there is a circuit in G that traverses every edge of the graph exactly once. If there is an open trail from vertex u to v in G and this trail traverses each edge in G exactly once, the trail is called Euler trail. [A trail from a vertex u to v is a path that does not involve a repeated edge] An Eulerian tour is a closed walk that starts at some vertex, passes through each edge exactly once and returns to the starting vertex.

Since any closed walk in an undirected graph enters and leaves any vertex the same number of times, the subgraph composed of the edges in any closed walk is even.Thus, if the graph contains a closed walk passing through each edge exactly once, the graph must be even, conversely,if the graph is even, then it contains an Euler tour,A path that passes through each edge exactly once but vertices may be repeated is called Euler path.

A graph that contains an Euler tour or Euler circuit is called an Eulerian graph.

Note:1.If a graph G has a vertex of odd degree,then there can be no Euler circuit in G.

2. If a graph G is connected and each vertex has even degree, then there is an Eulercircuit.

Forexample, the graph in figure5.23 is an Eulerian graph because all the vertices are of even degree, so it has an Euler circuit$v_1 \to v_2 \to v_3 \to v_5 \to v_4 \to v_3 \to v_1$. But the graph G in figure 5.24 is not an Euler graph because all the vertices of G are not even degrees, so there does not have any Euler circuit.

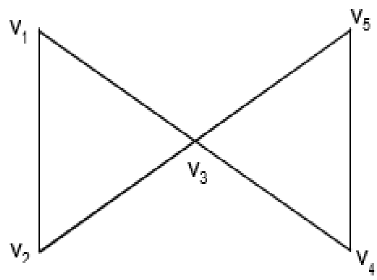


FIG 5.21 EULER GRAPH

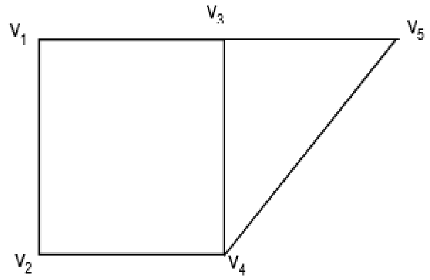


FIG 5.22 NOT EULER GRAPH

5.8.5 HAMILTONIAN GRAPH

Let G be a connected graph with more than two vertices. If there is a path in G that uses each vertex of the graph exactly once, then such a path is called Hamiltonian path. If the path is a circuit that contains each vertex in G exactly once, except the initial vertex, then such a path is called a Hamiltonian circuit. A graph that contains a **Hamiltonian circuit** is called a Hamiltoniangraph.

Note: i)The complete bipartite graph $k_{m,n}$is Hamiltonian if m = n and m>1.

ii) Eulerian circuit uses every edge exactly once but many repeat vertices, while Hamiltonian circuit uses each vertex exactly once except for the first and last vertex.

For example, the graph in figure 5.25 is Hamiltonian because there is a Hamiltonian circuit shown by the arrow symbols

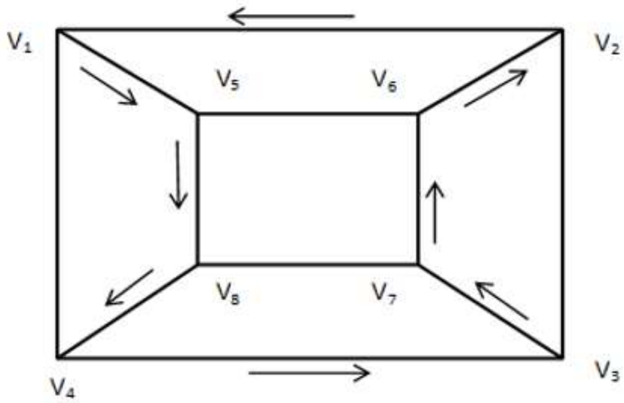


FIG 5.23 HAMILTONIAN GRAPH

Example: Draw three graph which are

1. Hamiltonian but not a Eulerian
2. Neither Eulerian nor Hamiltonian

Soln.1. Hamiltonian but not a Eulerian

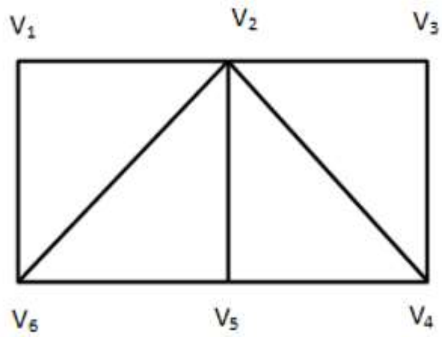


FIG 5.24

2 Neither Eulerian nor Hamiltonian

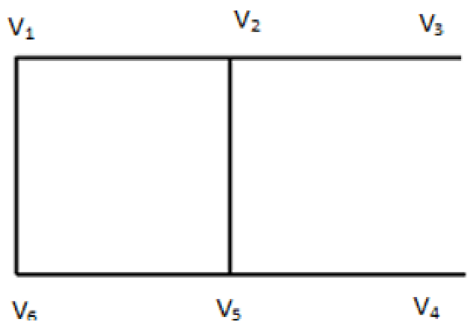


FIG 5.25

5.9 SUB GRAPHS

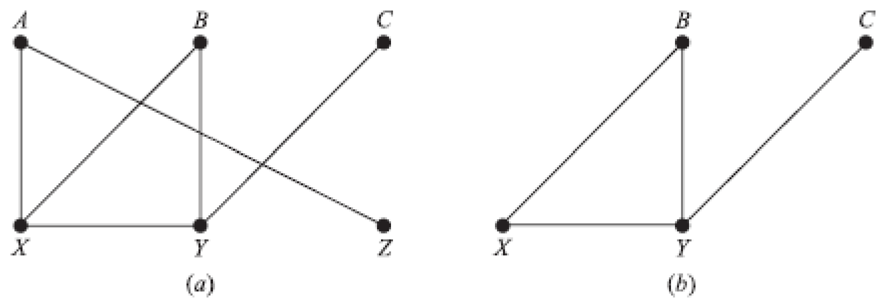
Consider a graph G = G (V,E). A graph H = H(V',E') is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if V' ⊆V and E' ⊆E. In particular:

(i) A subgraph H(V',E') of G(V,E) is called the subgraph induced by its vertices V' if its edge set E'contains all edges in G whose endpoints belong to vertices in H.

(ii) If v is a vertex in G, then G − v is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.

(iii) If e is an edge in G, then G − e is the subgraph of G obtained by simply deleting the edge e from G.

For example: In given figure Graph(b) is subgraph of Graph (a).



5.9.1 ISOMORPHISM OF GRAPHS

Two graphs are said to be isomorphic it they have identical behavior in terms of graph-theoretic properties. More precisely:

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two simple undirected graphs. A function f: $V_1 \rightarrow V_2$is called a graph isomorphism if

1. F is one-one and onto, i.e, there exists a one-to-one correspondence between their vertices as well as edges (both the graphs have equal number of vertices and edges, however, vertices may have different levels.)

2. for all $u,v \in V_1$, $\{u,v\} \in E_1$if and only if $\{f(u), f(v)\} \in E_2$

If such a function exists, then the graphs $G_1$and G_2 are called isomorphic graphs. For example,we can verify that the graph G and H in figure5.28 and are isomorphic

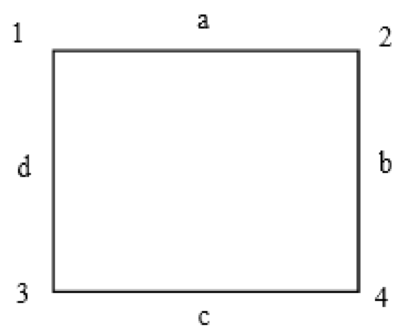


FIG 5.26

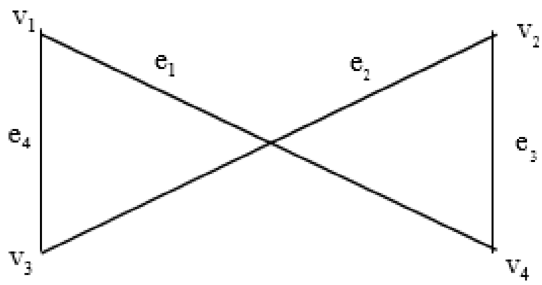


FIG 5.27

The correspondence between the two graphs is as follows:

The vertices 1, 2, 3 and 4 in G is corresponds to v_1, v_4, v_3 and v_2 respectively in

H. The edges a, b, c, d in G corresponds to e_1, e_3, e_2, e_4 respectively.

Note:i) Two isomorphic graphs have equal number of vertices and edges.

ii) Two isomorphic graphs have equal number of vertices withsamedegree.

Example 8: Show that the graph displayed in figure5.30 are not isomorphic.

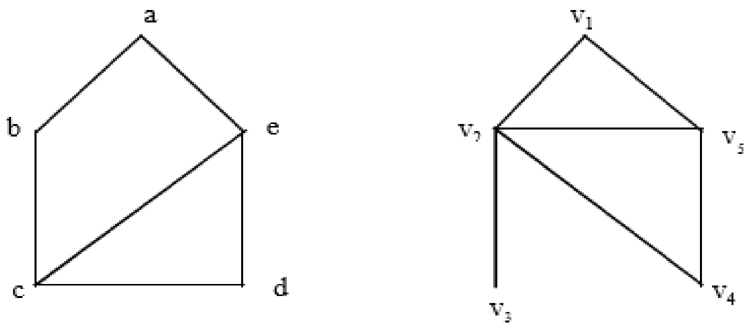
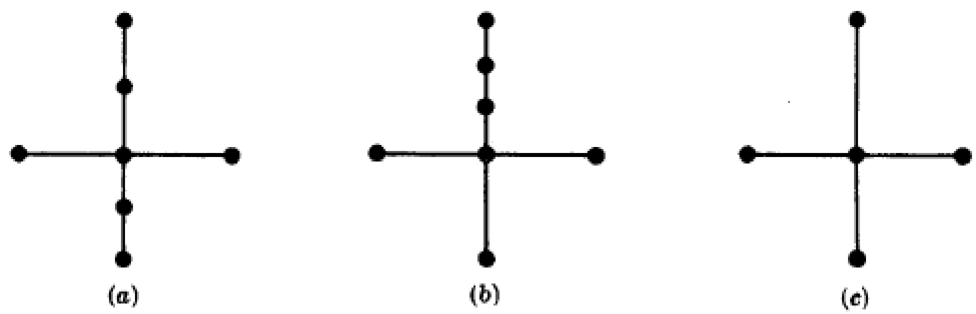


FIG 5.28

Solution: The graph G and H both have five vertices and six edges. However, the graph H has a vertex of degree one namely v3. Whereas G has no vertices of degree one. Hence G and H are not isomorphic.

5.9.2 Homeomorphic Graphs

Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G* are said to homeomorphic if they can be obtained from the same graph or isomorphic graphs by this method. The graphs (a) and (b) in given Fig. are not isomorphic, but they are homeomorphic since they can be obtained from the graph (c) by adding appropriate vertices.



Note: Graph (a) and (b) are Homeomorphic graphs

5.10 CUT POINT AND BRIDGES

Cut point and Bridges

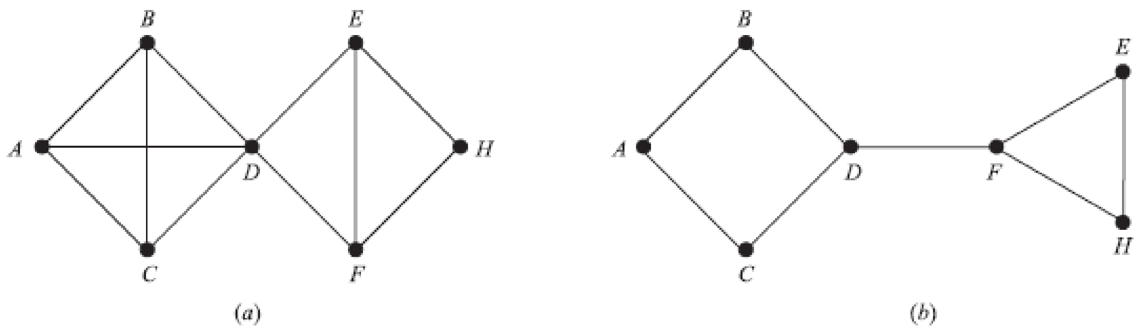
Let G be a connected graph. A vertex v in G is called a cut point if G−v is disconnected.

Note, that G−v is the graph obtained from G by deleting v and all edges containing v.

An edge e of G is called a bridge if G−e is disconnected.

Note, that G − e is the graph obtained from G by simply deleting the edge e.

For example: In given Fig.(a),the vertex D is a cutpoint and there are no bridges. In Fig.(b), the edge = {D,F} is a bridge. (Its endpoints D and F are necessarily cutpoints.)

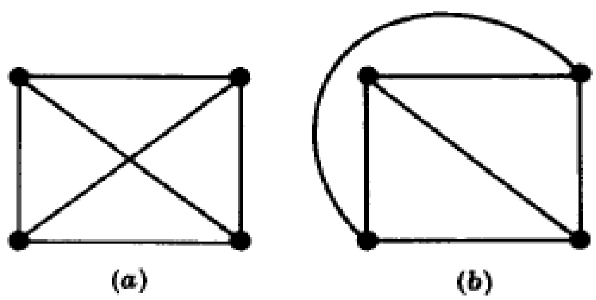


5.11 PLANAR GRAPHS

PLANAR GRAPHS

A graph or multigraph which can be drawn in the plane so that its edges do not cross is said to be *planar*.

Although the complete graph with four vertices *K*4 is usually pictured with crossing edges as in Fig. *(a)*, it can also be drawn with non crossing edges as in Fig.*(b)*; hence *K*4 is planar. Tree graphs form an important class of planar graphs.



Maps, Regions in planar graphs

A particular planar representation of a finite planar multigraph is called a map. We say that the map is connected if the underlying multigraph is connected. A given map divides the plane into various regions. For example, the map in below Fig. with six vertices and nine edges divides the plane into five regions. Observe that four of the regions are bounded, but the fifth region, outside the diagram, is unbounded. Thus there is no loss in generality in counting the number of regions if we assume that our map is contained in some large rectangle rather than in the entire plane. Observe that the border of each region of a map consists of edges. Sometimes the edges will form a cycle, but sometimes not. For example, in given Fig. the borders of all the regions are cycles except for r3. However, if we do move counterclockwise around r3 starting, say, at the vertex C, then we obtain the closed path (C,D,E, F,E,C) where the edge {*E,F*} occurs twice. By the *degree* of a region *r*, written deg*(r)*, we mean the length of the cycle or closed walk which borders *r*. We note that each edge either borders two regions or is contained in a region and will occur twice in any walk along the border of the region

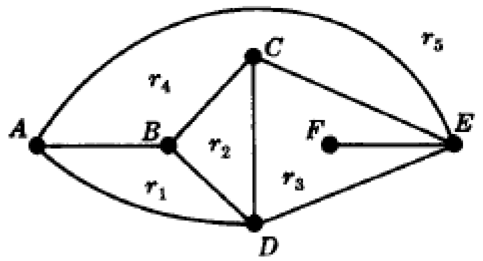


Fig.

Thus we have a theorem for regions which is analogous to Theorem for vertices.

Theorem: The sum of the degrees of the regions of a map is equal to twice the number of edges.

The degrees of the regions of above Fig. are:

deg$(r1)$ = 3, deg$(r2)$ = 3, deg$(r3)$ = 5, deg$(r4)$ = 4, deg$(r5)$ = 3

The sum of the degrees is 18, which, as expected, is twice the number of edges.

For notational convenience we shall picture the vertices of a map with dots or small circles, or we shall assume that any intersections of lines or curves in the plane are vertices.

5.11.1 Euler's Formula

Euler gave a formula which connects the number V of vertices, the number E of edges, and the number R of regions of any connected map. Specifically:

Theorem (Euler): V − E + R = 2.

Proof of Theorem:

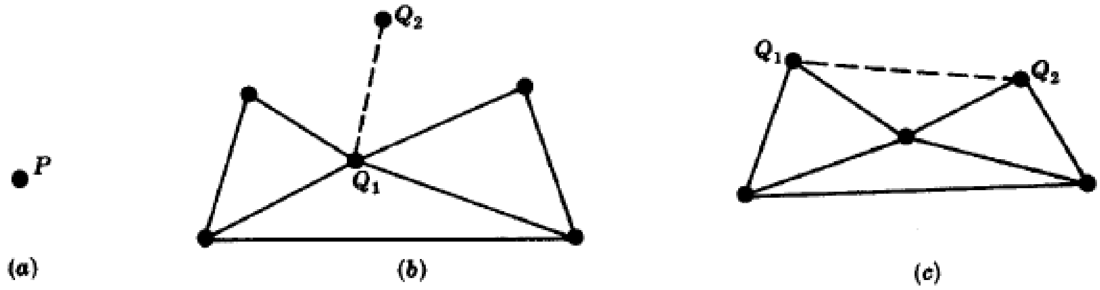
Suppose the connected map M consists of a single vertex P as in Fig. (a). Then V = 1, E = 0, and R = 1.

Hence V − E + R = 2. Otherwise M can be built up from a single vertex by the following two constructions:

(1) Add a new vertex Q2 and connect it to an existing vertex Q1 by an edge which does not cross any existing edge as in Fig. (b).

(2) Connect two existing vertices Q1 and Q2 by an edge e which does not cross any existing edge as in Fig.(c).

Neither operation changes the value of V −E +R. Hence M has the same value of V −E +R as the map consisting of a single vertex, that is, V − E + R = 2. Thus the theorem is proved.



For example: Observe that, in above given Fig., $V = 6$, $E = 9$, and $R = 5$; and, as expected by Euler's formula.

$V - E + R = 6 - 9 + 5 = 2$

It emphasize that the underlying graph of a map must be connected in order for Euler's formula to hold.

6.9CHECK YOUR PROGRESS

1. What is a simple graph?
2. Define degree of a vertex?
3. Define pendant and isolated vertex?
4. When a vertex is said to be an even or odd?
5. What do you mean by complete graph and regular graph?
6. Define adjacency matrix of undirected graph?
7. When two graphs are said to be isomorphic?
8. Define Eulerian path?

7.6.3 **6.10 SUMMARY**

1. A graph G = (v,E) is a pair of sets, where V ={v_1,v_2,.............. } is a set of vertices and E ={e_1,e_2,……} is a set of edges connecting pair of vertices.
2. A graph is said to be undirected graph if its edges are unordered pairs of distinct vertices otherwise the graph is said to be directed.
3. The degree of a vertex is the number of edges incident with that vertex.
4. A loop is an edge from a vertex to itself. It there is more than one edge between a pair of vertices, them these edges are called parallel edges.
5. A graph with no loops and parallel edges is called a simple graph.
6. A simple graph in which there is an edge between every pair of vertices is called a complete graph.
7. The number of vertices of odd degree in a graph is always even.
8. In computers, a graph can be represented in two ways, viz, adjacency matrix and incidence matrix.

9. The simple graph $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from v_1 to v_2 with the property that vertices a and b are adjacent in G_1 iff f(a)andf(b) are adjacent in G_2, $\forall a,b \in V_1$. Such a function f is called an isomorphism.
10. A graph is connected if we can reach any vertex from any other vertex by traveling along the edges. Otherwise, the graph is disconnected.
11. A graph is Hamiltonian if every vertex of the graph has even number of degrees.

6.11KEYWORDS

Graph: -A graph G = (v,E) is a pair of sets, where V = {v1, v2, ... } is a set of vertices and E = {e1, e2,……} is a set of edges connecting pair of vertices.

Simple graph: – A graph in which each edge connects two **different** vertices and where no two edges connect the same pair of vertices is called a simple graph.

Multigraph: – A graph in which multiple edges may connect the same pair of vertices is called a multigraph.

Complete Graphs: – A simple graph of n vertices having exactly one edge between each pair of vertices is called a complete graph. A complete graph of n vertices is denoted by K_n.

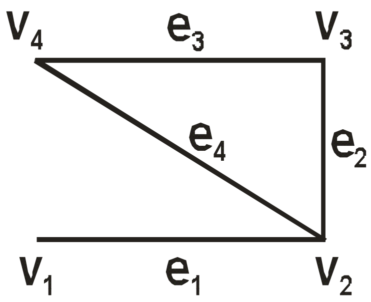
Bipartite Graphs: – A simple graph G is said to be bipartite if its vertex set V can be divided into two disjoint sets such that every edge in G has its initial vertex in the first set and the terminal vertex in the second set. Total number of edges are (n*m) with (n+m) vertices in bipartite graph.

Regular Graph: -A **Regular graph** is a graph in which degree of all the vertices is same. If the degree of all the vertices is k, then it is called k-regular graph.

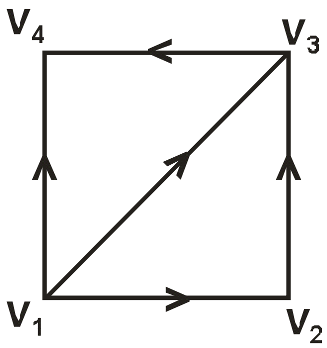
Planar graph: - A **planar graph** is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

6.12 SELF ASSESSMENT TEST

1. Write the adjacency matrix associated with the graph shown below:



2. Write incidence matrix of following graphs



3 Draw a graph corresponding to given adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

4. Draw the diagraph of the incidence matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

6.13 ANSWER TO CHECK YOURPROGRESS

1. A graph with no loops and parallel edges is called a simple graph.

2. The degree of a vertex is the number of edges incident with that vertex.

3. A vertex of degree one is called a pendant vertex and a vertex of degree zero is called an isolated vertex
4. A vertex is said to be an even or odd vertices according as its degree is an even or odd number.
5. A simple graph is which there exists an edge between every pair of vertices is called a complete graph. Again a graph in which every vertex has the same degree is called a regular graph.
6. Suppose that G be a simple undirected graph with n vertices. suppose that the vertices of G are listed arbitrarily as v1, v2, ..., vn. The adjacency matrix denoted by A(G) of G is ann×n matrix [aij] defined as:[aij] = 1, if vertex $v_i$is adjacent to v_j 0,otherwise
7. Two simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from $V_1$to$V_2$with the property that vertices a and b are adjacent in G_1 iff f(a) and f(b) are adjacent in G_2, $\forall a,b \in V_1$
8. A path that passes through each edge exactly once but vertices may be repeated is called on Euler path. Again a circuit that covers every edge exactly once is called an Euler circuit.

6.14 REFERENCES / SUGGESTED READINGS

1. Seymour Lepschutz, Finite mathematics (International edition 1983), McGraw-Hill Book Company, New York.
2. N.Deo, Graph Theory with application and computer science , Pentile H

SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
Course Code: MCA-25	AUTHOR: RENU BANSAL
Lesson No. 6	
WEIGHTED GRAPHS AND TREE	

STRUCTURE

6.1 LEARNINGOBJECTIVES

After going through this unit you will be able to know

1. What is weighted graphs.
2. How to apply dijkastra's shortest path algo.
3. What is tree graphs.
4. What is minimum spanning tree.
5. How to find minimum cost path between two nodes using prim's and kruskal algo.

6.2 INTRODUCTION

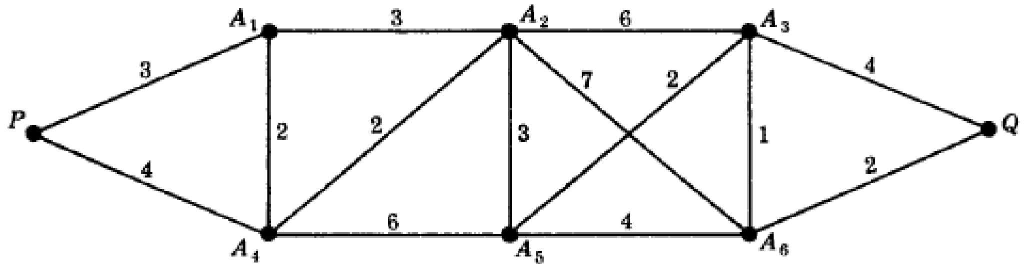
In Unit 5, we have discussed about various types of graphs and its applications. In this unit further on we discussed about what is weighted graphs, and various algos, which we can use in daily life for computing shortest path and minimum cost path between two nodes.

6.3 WEIGHTED GRAPH

A graph G is called a weighted graph if it's edges and/or vertices are assigned data of one kind or another. In particular, G is called a weighted graph if each edge e of G is assigned a non negative number w(e) called theweight or length of v. Below Figure shows a weighted graph where the weight of each edge is given in the obvious way. The weight (or length) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. One important problem in graph theory is to find a shortest path, that is, a path of minimum weight (length), between any two given vertices. The length of a shortest path between P and Q in Fig. is 14; one such path is

(P,A1,A2,A5,A3,A6,Q)

The reader can try to find another shortest path



6.3.1 Dijkstra's Algorithm

It is a greedy algorithm that solves the single-source shortest path problem for a directed graph G = (V, E) with nonnegative edge weights, i.e., w (u, v) ≥ 0 for each edge (u, v) ∈ E.

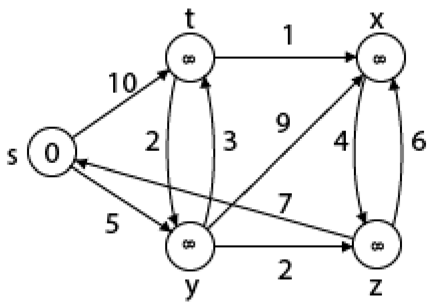
Dijkstra's Algorithm maintains a set S of vertices whose final shortest - path weights from the source s have already been determined. That's for all vertices v ∈ S; we have d [v] = δ (s, v). The algorithm repeatedly selects the vertex u ∈ V - S with the minimum shortest - path estimate, insert u into S and relaxes all edges leaving u.

Because it always chooses the "lightest" or "closest" vertex in V - S to insert into set S, it is called as the **greedy strategy**.

Dijkstra's Algorithm (G, w, s)

1. INITIALIZE - SINGLE - SOURCE (G, s)
2. S←∅
3. Q←V [G]
4. while Q ≠ ∅
5. do u ← EXTRACT - MIN (Q)
6. S ← S ∪ {u}
7. for each vertex v ∈ Adj [u]
8. do RELAX (u, v, w)

Example:



Solution:

Step1: Q =[s, t, x, y, z]

We scanned vertices one by one and find out its adjacent. Calculate the distance of each adjacent to the source vertices.

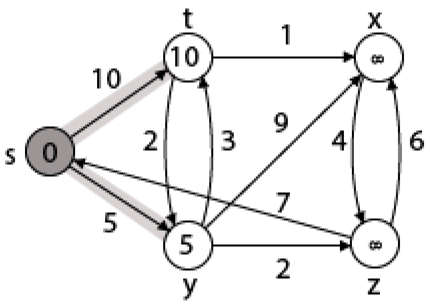
We make a stack, which contains those vertices which are selected after computation of shortest distance.

Firstly we take's' in stack M (which is a source)

1. M = [S] Q = [t, x, y, z]

Step 2: Now find the adjacent of s that are t and y.

1. Adj [s] → t, y [Here s is u and t and y are v]



Case - (i) s → t

d [v] > d [u] + w [u, v]

d [t] > d [s] + w [s, t]

∞ > 0 + 10 [false condition]

Then **d [t] ← 10**

π [t] ← 5

Adj [s] ← t, y

Case - (ii) s→ y

d [v] > d [u] + w [u, v]

d [y] > d [s] + w [s, y]

∞ > 0 + 5 [false condition]

∞ > 5

Then **d [y] ← 5**

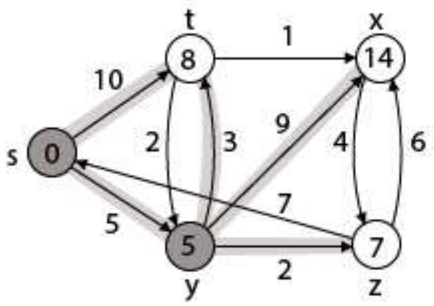
π [y] ← 5

By comparing case (i) and case (ii)

Adj [s] → t = 10, y = 5

y is shortest

y is assigned in 5 = [s, y]



Step 3: Now find the adjacent of y that is t, x, z.

1. Adj [y] → t, x, z [Here y is u and t, x, z are v]

Case - (i) y →t

d [v] > d [u] + w [u, v]

d [t] > d [y] + w [y, t]

10 > 5 + 3

10 > 8

Then d [t] ← 8

π [t] ← y

Case - (ii) y → x

d [v] > d [u] + w [u, v]

d [x] > d [y] + w [y, x]

∞ > 5 + 9

∞ > 14

Then d [x] ← 14

π [x] ← 14

Case - (iii) y → z

d [v] > d [u] + w [u, v]

d [z] > d [y] + w [y, z]

∞ > 5 + 2

∞ > 7

Then d [z] ← 7

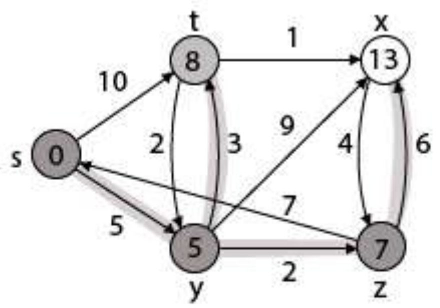
π [z] ← y

By comparing case (i), case (ii) and case (iii)

Adj [y] → x = 14, t = 8, z =7

z is shortest

z is assigned in 7 = [s, z]



Step - 4 Now we will find adj [z] that are s, x

1. Adj [z] → [x, s] [Here z is u and s and x are v]

Case - (i) z → x

d [v] > d [u] + w [u, v]

d [x] > d [z] + w [z, x]

14 > 7 + 6

14 > 13

Then d [x] ← 13

π [x] ← z

Case - (ii) z → s

d [v] > d [u] + w [u, v]

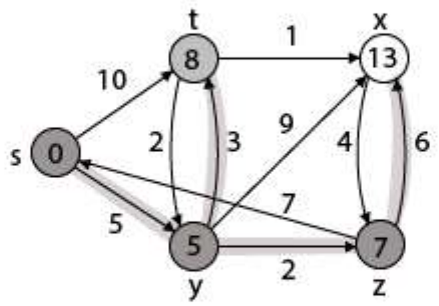
d [s] > d [z] + w [z, s]

0 > 7 + 7

0 > 14

∴ This condition does not satisfy so it will be discarded.

Now we have x = 13.



Step 5: Now we will find Adj [t]

Adj [t] → [x, y] [Here t is u and x and y are v]

Case - (i) t → x

d [v] > d [u] + w [u, v]

d [x] > d [t] + w [t, x]

13 > 8 + 1

13 > 9

Then **d [x] ← 9**

π [x] ← t

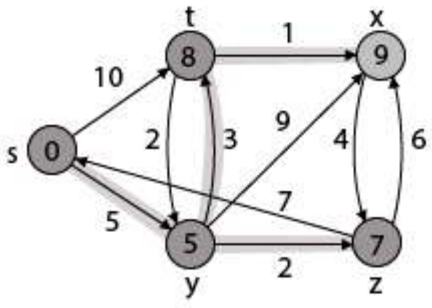
Case - (ii) t → y

d [v] > d [u] + w [u, v]

d [y] > d [t] + w [t, y]

5 > 10

∴ This condition does not satisfy so it will be discarded.



Thus we get all shortest path vertex as

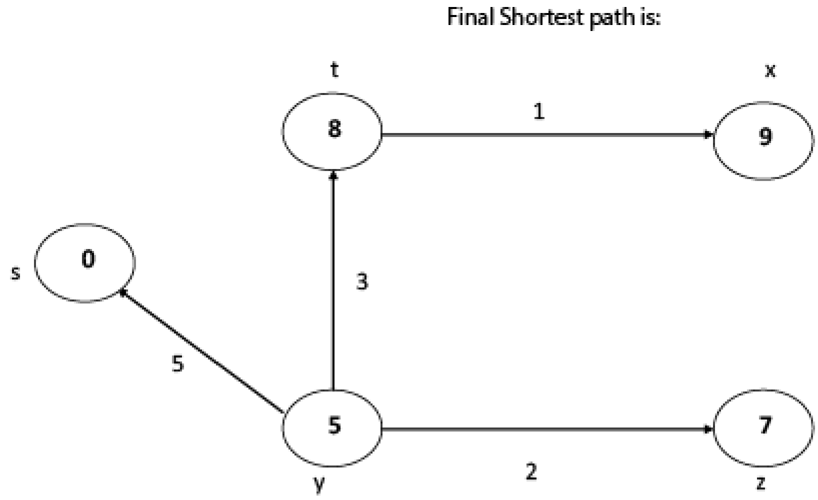
Weight from s to y is 5

Weight from s to z is 7

Weight from s to t is 8

Weight from s to x is 9

These are the shortest distance from the source's' in the given graph.



Disadvantage of Dijkstra's Algorithm:

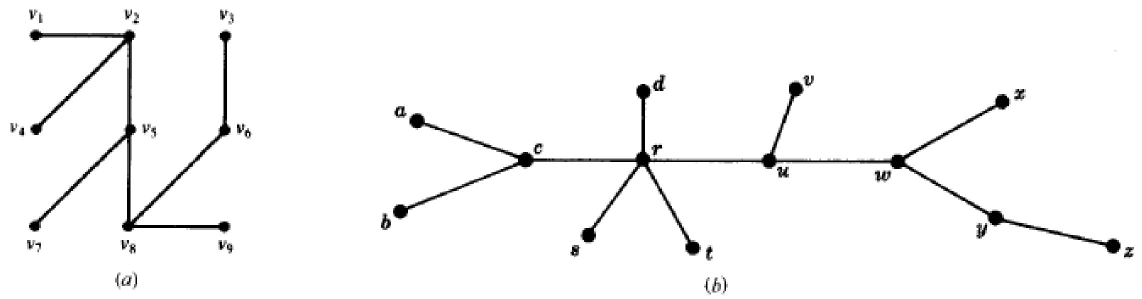
1. It does a blind search, so wastes a lot of time while processing.
2. It can't handle negative edges.
3. It leads to the acyclic graph and most often cannot obtain the right shortest path.
4. We need to keep track of vertices that have been visited.

6.4 Tree

A graph T is called a tree if T is connected and T has no cycles. Examples of trees are shown in Fig. A forest G is a graph with no cycles; hence the connected components of a forest G are trees. A graph without cycles is said to be cycle-free. The tree consisting of a single vertex with no edges is called the degenerate tree. Consider a tree T . Clearly, there is only one simple path between two vertices of T ; otherwise, the two paths would form a cycle. Also:

(a) Suppose there is no edge {u, v} in T and we add the edge e = {u, v} to T . Then the simple path from u to v in T and e will form a cycle; hence T is no longer a tree.

(b) On the other hand, suppose there is an edge e = {u, v} in T , and we delete e from T . Then T is no longer connected (since there cannot be a path from u to v); hence T is no longer a tree.



The following theorem applies when our graphs are finite.

Theorem : Let G be a graph with n >1 vertices. Then the following are equivalent:

(i) G is a tree.

(ii) G is a cycle-free and has n − 1 edges.

(iii) G is connected and has n − 1 edges.

Proof:

The proof is by induction on *n*. The theorem is certainly true for the graph with only one vertex and hence no edges. That is, the theorem holds for $n = 1$.We now assume that $n > 1$ and that the theorem holds for graphs with less than *n* vertices.

(i) *implies* (ii) Suppose *G* is a tree. Then *G* is cycle-free, so we only need to show that *G* has $n-1$ edges. *G* has a vertex of degree 1. Deleting this vertex and its edge, we obtain a tree *T* which has $n - 1$ vertices. The theorem holds for *T* , so *T* has $n - 2$ edges. Hence *G* has $n - 1$ edges. (ii) *implies* (iii) Suppose *G* is cycle-free and has $n - 1$ edges. We only need show that *G* is connected. Suppose *G* is disconnected and has *k* components, $T1, \dots, Tk$, which are trees since each is connected and cycle-free. Say *Ti* has *ni* vertices. Note $ni < n$. Hence the theorem holds for *Ti*, so *Ti* has $ni - 1$ edges. Thus

$$n = n1 + n2 + \cdots + nk$$

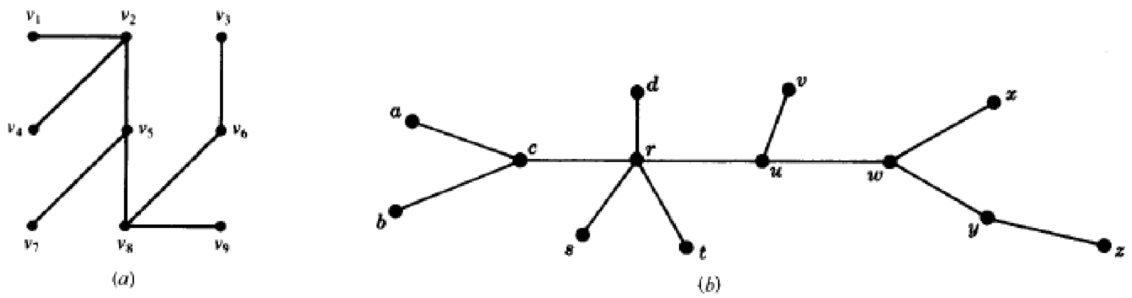
and

$$n - 1 = (n1 - 1) + (n2 - 1) + \cdots + (nk - 1) = n1 + n2 + \cdots + nk - k = n - k$$

Hence k = 1. But this contradicts the assumption that G is disconnected and has k >1 components. Hence G is connected.

(iii) *implies* (i) Suppose G is connected and has $n - 1$ edges. We only need to show that G is cycle-free. Suppose G has a cycle containing an edge e. Deleting e we obtain the graph $H = G - e$ which is also connected. But H has n vertices and $n - 2$ edges, and this contradicts. Thus G is cycle-free and hence is a tree.

This theorem also tells us that a finite tree T with n vertices must have $n-1$ edges. For example, the tree shown in below Fig.(a)has 9 vertices and 8 edges, and the tree in Fig.*(b)* has 13 vertices and 12 edges.

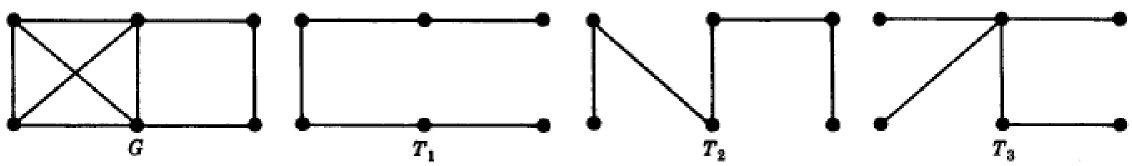


6.5 SPANNING TREES

6.5.1 Spanning Trees

A subgraph T of a connected graph G is called a *spanning tree* of G if T is a tree and T includes all the vertices of G. Figure shows a connected graph G and spanning trees $T1$, $T2$, and $T3$ of G.

6.5.2 Minimum Spanning Trees



Suppose G is a connected weighted graph. That is, each edge of G is assigned a nonnegative number called the weight of the edge. Then any spanning tree T of G is assigned a total weight obtained by adding the weights of the edges in T . A minimal spanning tree (MST) of G is a spanning tree whose total weight is as small as possible.

NOTE:

The weight of a minimal spanning tree is unique, but the minimal spanning tree itself is not. Different minimal spanning trees can occur when two or more edges have the same weight. In such a case, the arrangement of the edges in tree is not unique and hence may result in different minimal spanning trees.

6.5.1 Methods of Minimum Spanning Tree

There are two methods to find Minimum Spanning Tree

1. Kruskal's Algorithm
2. Prim's Algorithm

6.5.2 Kruskal's Algorithm:

It is an algorithm to construct a Minimum Spanning Tree (MST) for a connected weighted graph. It is a Greedy Algorithm. The Greedy Choice is to put the smallest weight edge that does not because a cycle in the Minimum Spanning Tree constructed so far.

Steps for finding MST using Kruskal's Algorithm:

1. Arrange the edge of G in order of increasing weight.
2. Starting only with the vertices of G and proceeding sequentially add each edge which does not result in a cycle, until (n - 1) edges are used.
3. EXIT.

MST- KRUSKAL (G, w)

1. A ← ∅
2. for each vertex v ∈ V [G]
3. do MAKE - SET (v)
4. sort the edges of E into non decreasing order by weight w

5. for each edge (u, v) ∈ E, taken in non decreasing order by weight

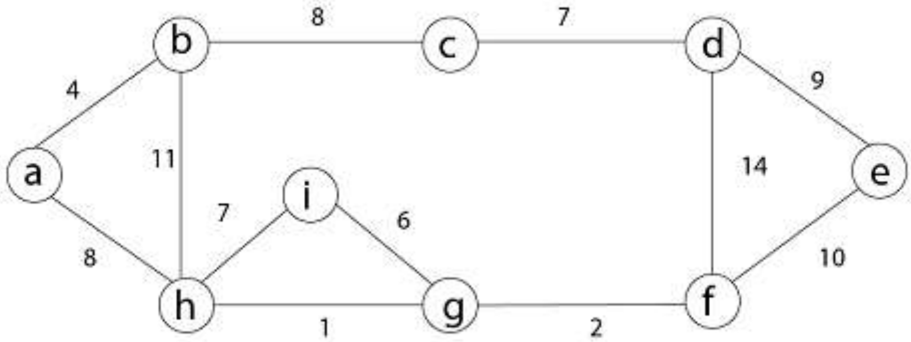
6. do if FIND-SET (μ) ≠ if FIND-SET (v)

7. then A ← A ∪ {(u, v)}

8. UNION (u, v)

9. return A

For Example: Find the Minimum Spanning Tree of the following graph using Kruskal's algorithm.



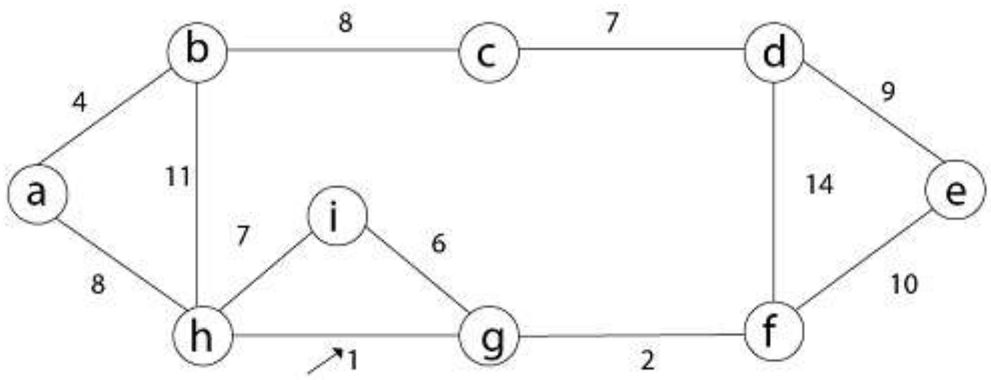
Solution: First we initialize the set A to the empty set and create |v| trees, one containing each vertex with MAKE-SET procedure. Then sort the edges in E into order by non-decreasing weight.

There are 9 vertices and 12 edges. So MST formed (9-1) = 8 edges

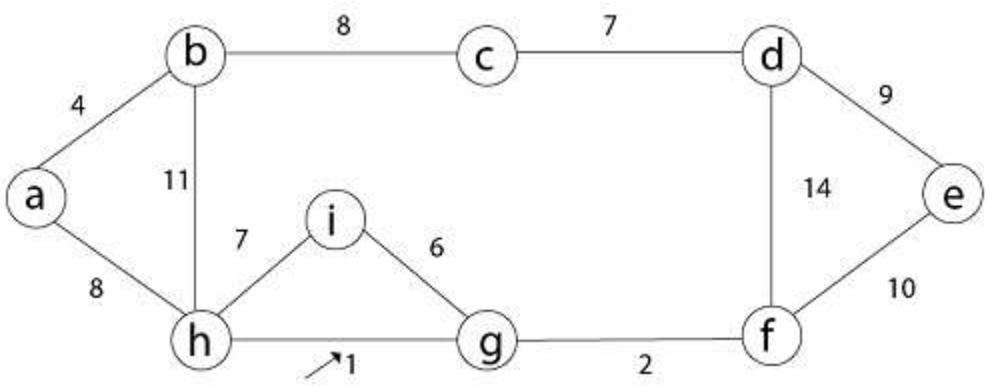
Weight	Source	Destination
1	h	g
2	g	f
4	a	b
6	i	g
7	h	i
7	c	d
8	b	c
8	a	h
9	d	e
10	e	f
11	b	h
14	d	f

Now, check for each edge (u, v) whether the endpoints u and v belong to the same tree. If they do then the edge (u, v) cannot be supplementary. Otherwise, the two vertices belong to different trees, and the edge (u, v) is added to A, and the vertices in two trees are merged in by union procedure.

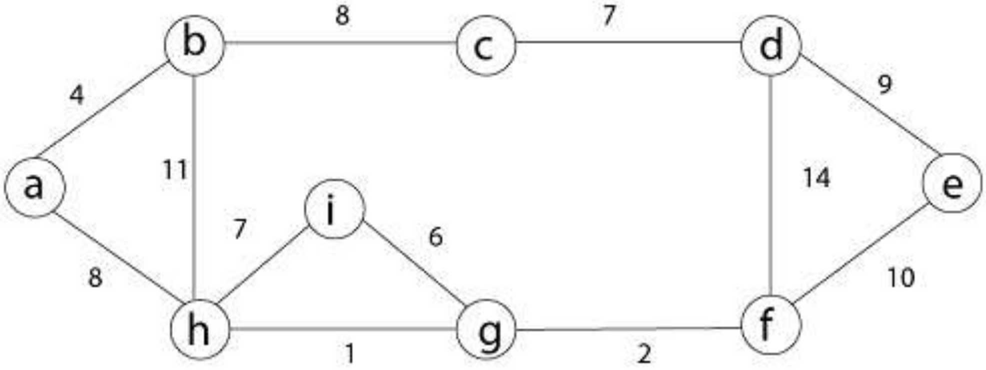
Step1: So, first take (h, g) edge



Step 2: then (g, f) edge.

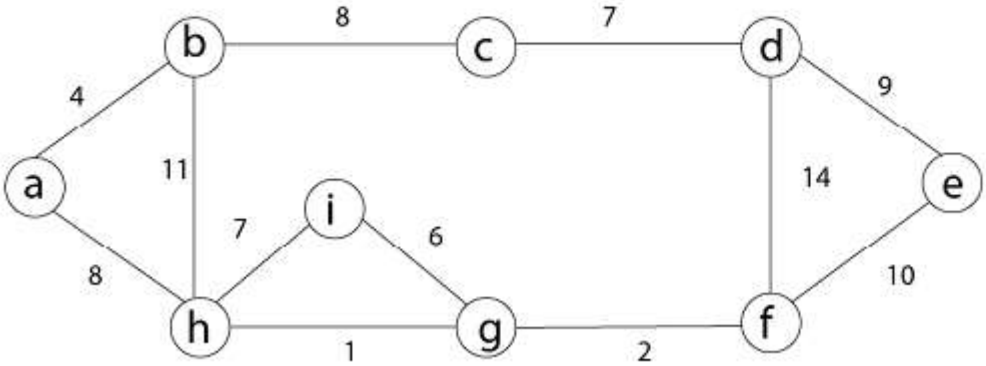


Step 3: then (a, b) and (i, g) edges are considered, and the forest becomes



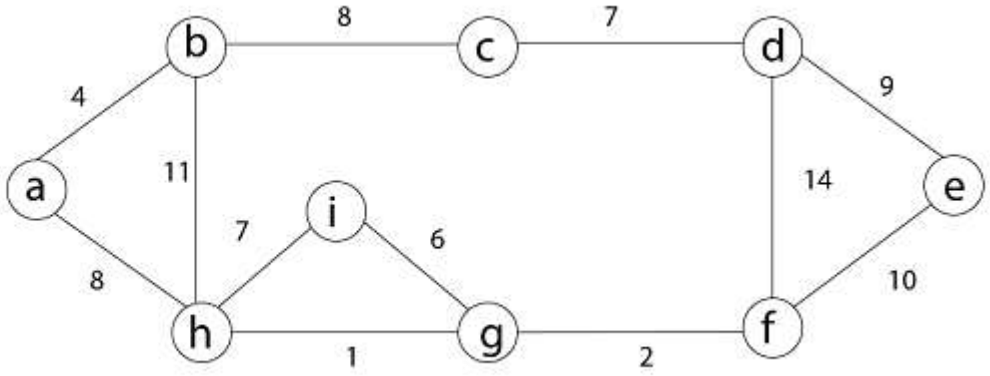
Step 4: Now, edge (h, i). Both h and i vertices are in the same set. Thus it creates a cycle. So this edge is discarded.

Then edge (c, d), (b, c), (a, h), (d, e), (e, f) are considered, and the forest becomes.



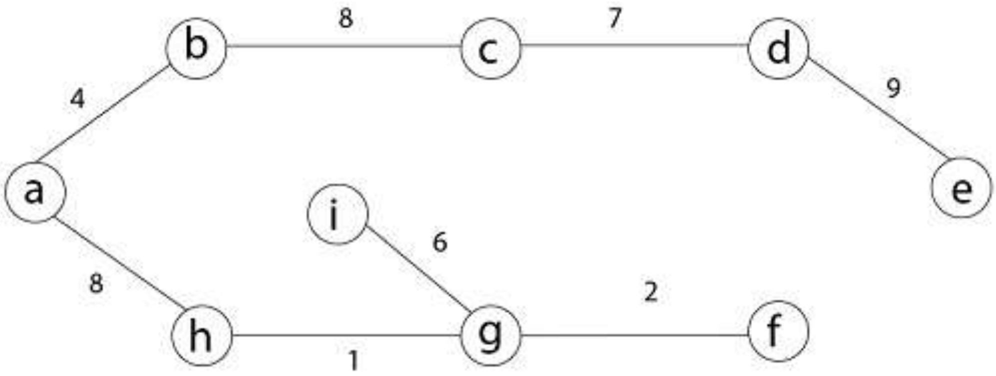
Step 5: In (e, f) edge both endpoints e and f exist in the same tree so discarded this edge. Then (b, h) edge, it also creates a cycle.

Step 6: After that edge (d, f) and the final spanning tree is shown as in dark lines.



Step 7: This step will be required Minimum Spanning Tree because it contains all the 9 vertices and (9 - 1) = 8 edges

1. e → f, b → h, d → f [cycle will be formed]



Minimum Cost MST

6.5.3 Prim's Algorithm

It is also a greedy algorithm. It starts with an empty spanning tree. The idea is to maintain two sets of vertices:

- Contain vertices already included in MST.
- Contain vertices not yet included.

At every step, it considers all the edges and picks the minimum weight edge. After picking the edge, it moves the other endpoint of edge to set containing MST.

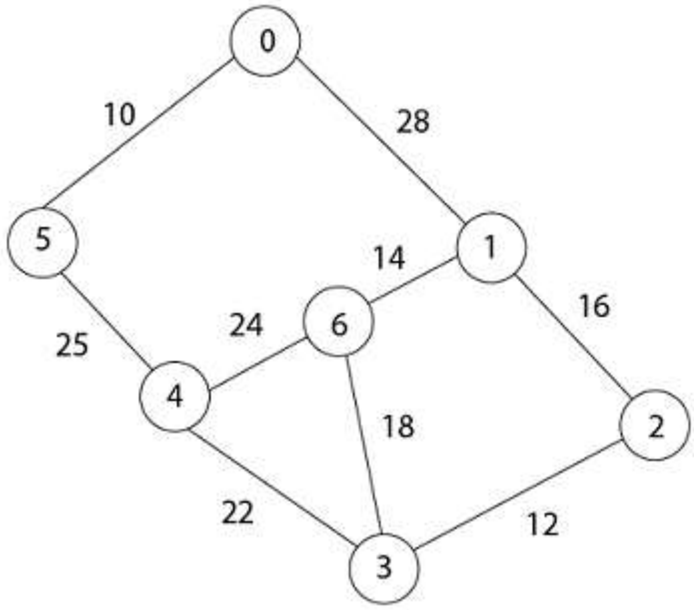
7.6.4 Steps for finding MST using Prim's Algorithm:

1. Create MST set that keeps track of vertices already included in MST.
2. Assign key values to all vertices in the input graph. Initialize all key values as INFINITE (∞). Assign key values like 0 for the first vertex so that it is picked first.
3. While MST set doesn't include all vertices.
    a. Pick vertex u which is not is MST set and has minimum key value. Include 'u'to MST set.
    b. Update the key value of all adjacent vertices of u. To update, iterate through all adjacent vertices. For every adjacent vertex v, if the weight of edge u.v less than the previous key value of v, update key value as a weight of u.v.

MST-PRIM (G, w, r)

1. for each u ∈ V [G]
2. do key [u] ← ∞
3. π [u] ← NIL
4. key [r] ← 0
5. Q ← V [G]
6. While Q ? ∅
7. do u ← EXTRACT - MIN (Q)
8. for each v ∈ Adj [u]
9. do if v ∈ Q and w (u, v) < key [v]
10. then π [v] ← u
11. key [v] ← w (u, v)

Example: Generate minimum cost spanning tree for the following graph using Prim's algorithm.



Solution: In Prim's algorithm, first we initialize the priority Queue Q. to contain all the vertices and the key of each vertex to ∞ except for the root, whose key is set to 0. Suppose 0 vertex is the root, i.e., r. By EXTRACT - MIN (Q) procure, now u = r and Adj [u] = {5, 1}.

Removing u from set Q and adds it to set V - Q of vertices in the tree. Now, update the key and π fields of every vertex v adjacent to u but not in a tree.

Vertex	0	1	2	3	4	5	6
Key Value	0	∞	∞	∞	∞	∞	∞
Parent	NIL	NIL	NIL	NIL	NIL	NIL	NIL

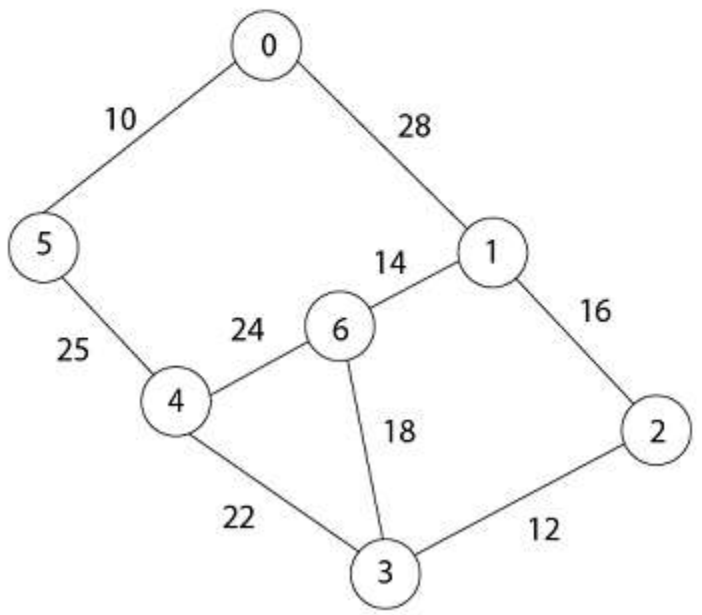
1. Taking 0 as starting vertex
2. Root = 0
3. Adj [0] = 5, 1
4. Parent, π [5] = 0 and π [1] = 0
5. Key [5] = ∞ and key [1] = ∞
6. w [0, 5) = 10 and w (0,1) = 28

7. w (u, v) < key [5] , w (u, v) < key [1]

8. Key [5] = 10 and key [1] = 28

9. So update key value of 5 and 1 is:

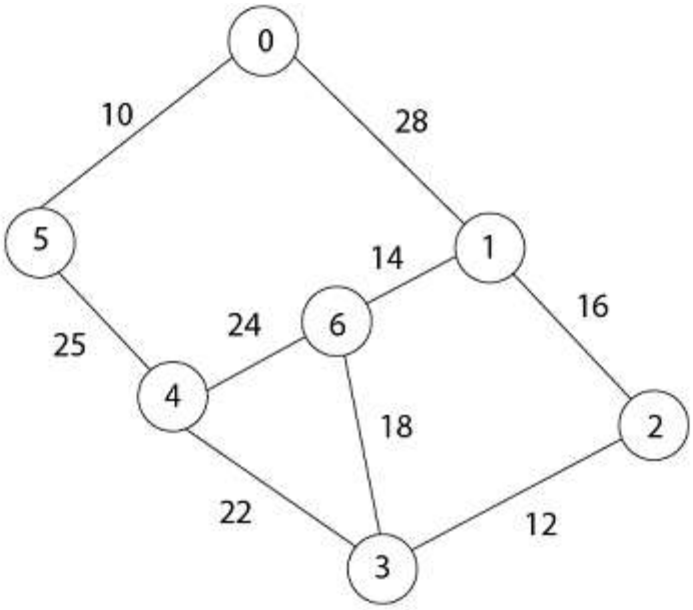
Vertex	0	1	2	3	4	5	6
Key Value	0	28	∞	∞	∞	10	∞
Parent	NIL	0	NIL	NIL	NIL	0	NIL



Now by EXTRACT_MIN (Q) Removes 5 because key [5] = 10 which is minimum so u = 5.

1. Adj [5] = {0, 4} and 0 is already in heap
2. Taking 4, key [4] = ∞ π [4] = 5
3. (u, v) < key [v] then key [4] = 25
4. w (5,4) = 25
5. w (5,4) < key [4]
6. date key value and parent of 4.

Vertex	0	1	2	3	4	5	6
Key Value	0	28	∞	∞	25	10	∞
Parent	NIL	0	NIL	NIL	5	0	NIL



Now remove 4 because key [4] = 25 which is minimum, so u =4

1. Adj [4] = {6, 3}
2. Key [3] = ∞ key [6] = ∞
3. w (4,3) = 22 w (4,6) = 24
4. w (u, v) < key [v] w (u, v) < key [v]
5. w (4,3) < key [3] w (4,6) < key [6]

Update key value of key [3] as 22 and key [6] as 24.

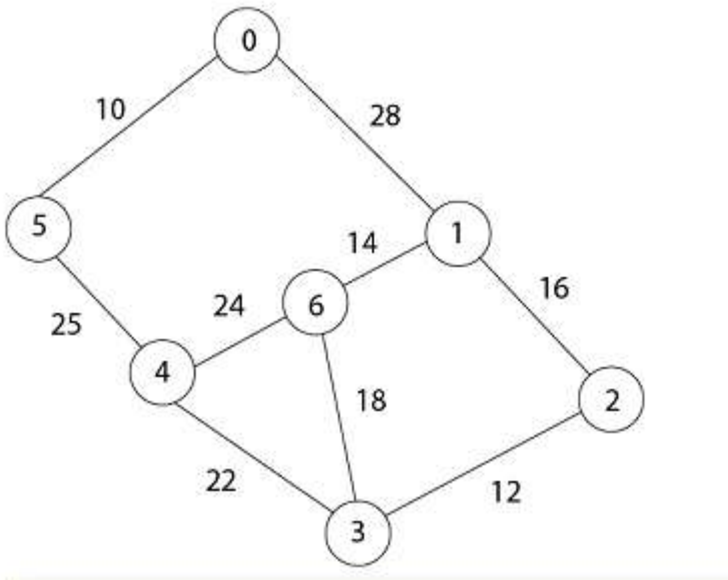
And the parent of 3, 6 as 4.

1. π[3]= 4 π[6]= 4

Vertex	0	1	2	3	4	5	6
Key Value	0	28	∞	22	25	10	24
Parent	NIL	0	NIL	4	5	0	4

1. u = EXTRACT_MIN (3, 6) [key [3] < key [6]]
2. u = 3 i.e. 22 < 24

Now remove 3 because key [3] = 22 is minimum so u =3.



1. Adj [3] = {4, 6, 2}
2. 4 is already in heap
3. 4 ≠ Q key [6] = 24 now becomes key [6] = 18
4. Key [2] = ∞ key [6] = 24
5. w (3, 2) = 12 w (3, 6) = 18
6. w (3, 2) < key [2] w (3, 6) < key [6]

Now in Q, key [2] = 12, key [6] = 18, key [1] = 28 and parent of 2 and 6 is 3.

1. π [2] = 3 π[6]=3

Now by EXTRACT_MIN (Q) Removes 2, because key [2] = 12 is minimum.

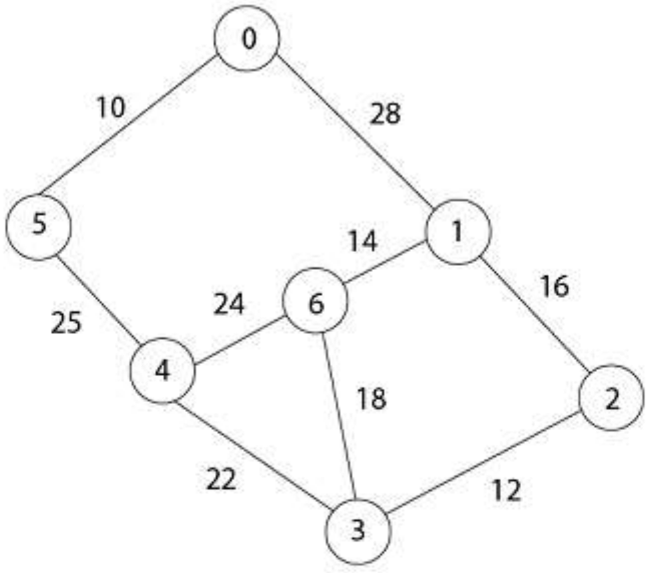
Vertex	0	1	2	3	4	5	6
Key Value	0	28	12	22	25	10	18
Parent	NIL	0	3	4	5	0	3

1. u = EXTRACT_MIN (2, 6)
2. u = 2 [key [2] < key [6]]
3. 12 < 18
4. Now the root is 2
5. Adj [2] = {3, 1}
6. 3 is already in a heap
7. Taking 1, key [1] = 28
8. w (2,1) = 16
9. w (2,1) < key [1]

   So update key value of key [1] as 16 and its parent as 2.

1. $\pi[1] = 2$

Vertex	0	1	2	3	4	5	6
Key Value	0	16	12	22	25	10	18
Parent	NIL	2	3	4	5	0	3



Now by EXTRACT_MIN (Q) Removes 1 because key [1] = 16 is minimum.

1. Adj [1] = {0, 6, 2}
2. 0 and 2 are already in heap.
3. Taking 6, key [6] = 18
4. w [1, 6] = 14
5. w [1, 6] < key [6]

Update key value of 6 as 14 and its parent as 1.

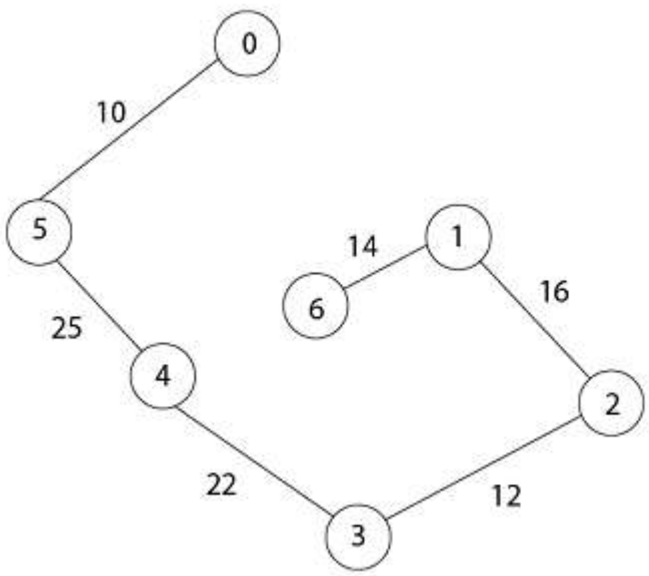
1. Π [6] = 1

Vertex	0	1	2	3	4	5	6
Key Value	0	16	12	22	25	10	14
Parent	NIL	2	3	4	5	0	1

Now all the vertices have been spanned, Using above the table we get Minimum Spanning Tree.

1. 0 → 5 → 4 → 3 → 2 → 1 → 6
2. [Because Π [5] = 0, Π [4] = 5, Π [3] = 4, Π [2] = 3, Π [1] =2, Π [6] =1]

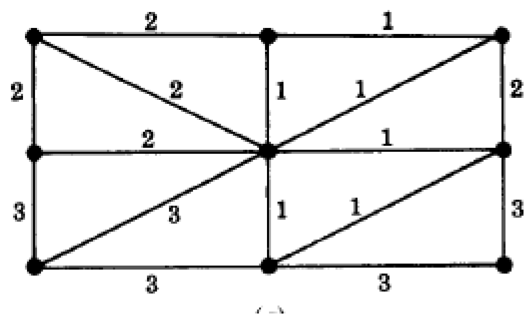
Thus the final spanning Tree is



Total Cost = 10 + 25 + 22 + 12 + 16 + 14 = 99

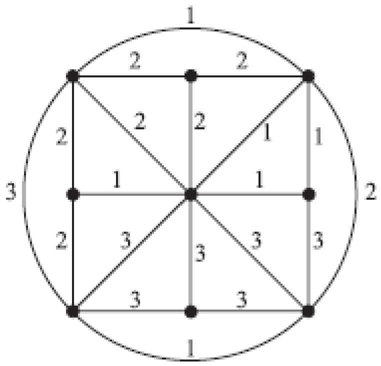
6.6 CHECK YOUR PROGRESS

1. Find a minimal spanning tree T for the weighted graph G in figure below?

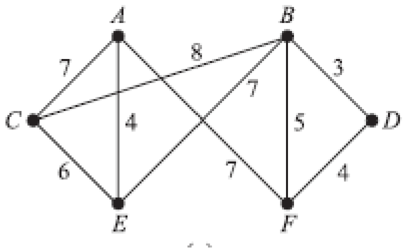


2. Find the number of trees with seven vertices.

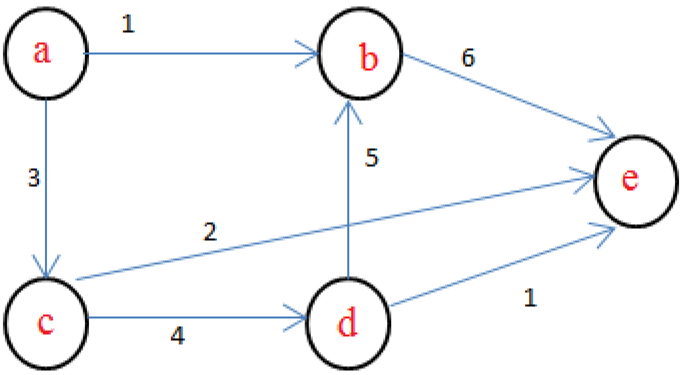
3. Find the weight of a minimum spanning tree by prim's algorithm. in Fig.



4. Use kruskal's algorithm to obtain minimum spanning tree for following figure?



5. In the given graph, identify the shortest path using Dijkastra's algorithm having minimum cost to reach vertex E if A is the source vertex.



6.7SUMMARY

1. Graph without cycle is known as tree.
2. When we give weights or label on edeges of any graph , then the graph is known as weighted graph.

3. Dijkastra's Algorithm shows the minimum weight path between two nodes.

4. Two minimum spanning tree for the same graph have unique weight but can be of different structure.

5. Kruskal's algo and prim's algo are two algorithms to find minimum spanning tree.

6.8 KEYWORDS

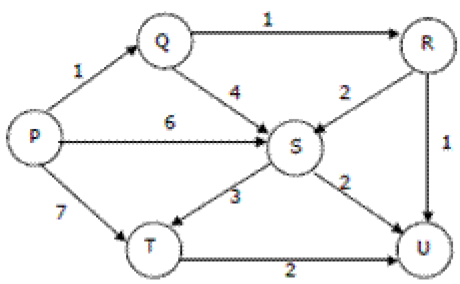
Weighted graph – A graph G is called a weighted graph if its edges and/or vertices are assigned data of one kind or another.

Trees –non cyclic structure of graph is known as tree.

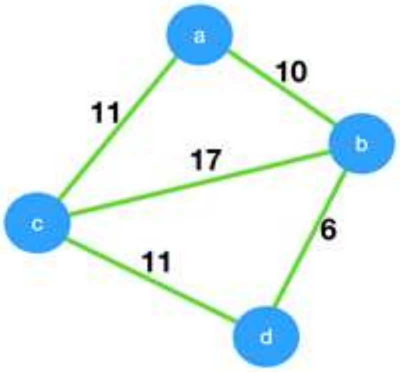
Minimum Spanning Tree – A minimal spanning tree (MST) of G is a spanning tree whose total weight is as small as possible.

6.9SELF ASSESSMENT TEST

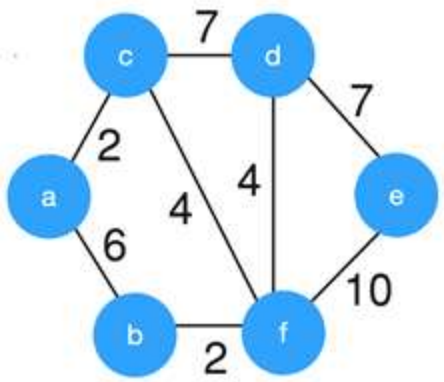
Q.1 Suppose we run Dijkstra's single source shortest-path algorithm on the following edge weighted directed graph with vertex P as the source. In what order do the nodes get included into the set of vertices for which the shortest path distances are finalized?



Q.2 What is the weight of the minimum spanning tree using the Prim's algorithm, starting from vertex a?

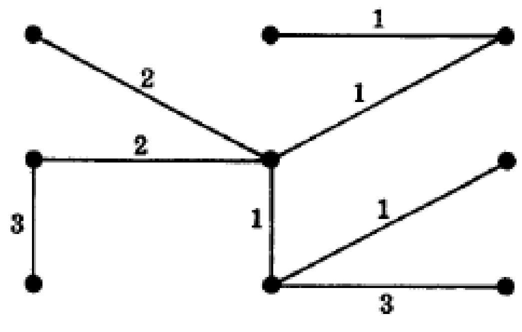


Q.3 What is the weight of the minimum spanning tree using the Kruskal's algorithm?



6.10 ANSWERS TO CHECK YOUR PROGRESS

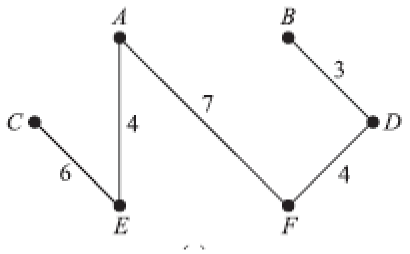
Ans: 1 Since G has $n = 9$ vertices, T must have $n - 1 = 8$ edges. Apply Algorithm, that is, keep deleting edges with maximum length and without disconnecting the graph until only $n-1 = 8$ edges remain and without forming any circle until $n - 1 = 8$ edges are added. Now we got a minimum spanning tree such as that shown in Fig.



Ans:2 10

Ans:3 1 + 1 + 1 + 1 + 1 + 2 + 2 + 3 = 12.

Ans:4



Thus the minimal spanning tree of *Q* which is obtained contains the edges

BD, *AE*, *DF*, *CE*, *AF*

The spanning tree appears in Fig. it has weight 24.

Ans:5 The minimum cost required to travel from vertex A to E is via vertex C

A to C, cost=3

C to E, cost=2

Total Cost=5, hence Ans is a-c-e.

6.11REFERENCES / SUGGESTED READINGS

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
COURSE CODE: MCA-25	AUTHOR: RENU BANSAL
LESSON NO. 7	
INTRODUCTION TO OPERATIONS RESEARCH AND TECHNIQUES	

STRUCTURE

7.1 LEARNINGOBJECTIVE

THE MAIN OBJECTIVE OF THIS LESSON IS TO ACQUAINT THE STUDENTS WITH THE CONCEPT OF OPERATIONS RESEARCH, ITS HISTORICAL DEVELOPMENT, ITS CHARACTERISTICS AND SCOPES IN DIFFERENT FIELDS. MODELS USED IN OPERATIONS RESEARCH AND THE METHODS TO SOLVE THESE MODELS ARE ALSO EXPLAINED.

7.2 ORIGIN & DEVELOPMENT OF OPERATIONS RESEARCH

The main origin of Operations Research (OR) was during the Second World-War. At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defense of the country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, e.g. the efficient ocean transport, effective bombing, etc.

During World-War II, the Military Commands of U.K. and U.S. A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and plans for adding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The OR teams were not actually engaged in military operations and in fighting the war. But they were only advisors and significantly instrumental in winning the war to the extent that the scientific and systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with **“an art of winning the war without actually fighting it”.**

As the name implies, ‘Operations Research’ (sometime abbreviated OR) was apparently invented because the team was dealing with research on (military) operations. The work of this team of scientists was named as Operational Research in England.

The encouraging result obtained by the British OR teams quickly motivated the United States military management to start with similar activities. Successful application of the U.S. teams included the invention of new fight patterns, planning sea mining and effective utilization of electronic equipment.

Following the end of war, the success of military teams attracted the attention of Industrial managers who were seeking solutions to their complex executive-type problems. The most common problem was: what methods should be adopted so that the total cost is minimum or total profits maximum? The first mathematical model in this field (called the Simplex Method of linear programming) was developed in 1947 by American mathematician, **George B. Dantzig**. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.

7.3 DEFINITIONS OF OPERATIONS RESEARCH

'OR' has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. So it is important and interesting to give below a few opinions about the definition of OR which have been changed according to the development of the subject:

1. OR is a scientific method of providing executive department with a quantitative basis for decision regarding the operations under their control.

   -Morse and Kimbal (1946)

2. OR is a scientific method of providing executive with an analytical and objective basis for decisions. **-P.M.S. Blackett (1948)**

3. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide these in control of the operations with optimum solutions to the problem.

   -Churchman, Acoff, Arnoff (1957)

4. OR is the art to giving bad answers to problems to which otherwise worse answers are given. **-T.L. Saaty (1958)**

5. Operations Research is the art of winning war without actually fighting it.

6. OR is a scientific approach to problem solving for executive management.
-H.M. Wagner

Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasis its only one or the other aspect.

7.4 NATURE AND CHARACTERISTICS FEATURES OF O.R

Operations Research has the following characteristics:

1. **Inter-disciplinary team approach-** In OR, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

2. **Wholistic approach to the system-** The most of the problems tackled by OR have the characteristic that OR tries to find the best (optimum)decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

3. **Imperfectness of solutions-** By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

4. **Use of scientific research-** OR uses techniques of scientific research to find the optimum solution.

5. **To optimize the total output-** OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

7.5 SCOPE/ APPLICATIONS OF OPERATIONS RESEARCH

Operations Research has scopes in almost every field. Applications of Operations Research in various fields are:

1. **In Agriculture-** Optimum allocation of land to various crops in accordance with the climatic conditions; and optimum distribution of water from various resources like canal for irrigation purposes.
2. **In Finance-** OR- Techniques can be fruitfully applied:
   i. To maximize the per capita income with minimum resources;
   ii. Ti find out the profit plan for the company;
   iii. To determine the best replacement policies, etc.
3. **In Industry-** OR is useful to the Industry Director in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc., to arrive at the optimum decision.
4. **In Marketing-** With the help of OR techniques a Marketing Administrator (Manager) can decide:
   i. Where to distribute the products for sale so that the total cost of transportation etc. is minimum,
   ii. The minimum per unit sale price.
   iii. The size of the stock to meet the future demand.
   iv. How to select the best advertizing media with respect to time, cost, etc.
   v. How, when, and what to purchase at the minimum possible cost?
5. **In Personal Management.** A personal manager can use OR techniques:
   i. To appoint the most suitable persons on minimum salary.
   ii. To determine the best age of retirement for the employees.

iii. To find out the number of persons to be appointed on full time basis when the workload is seasonal.

6. **In Production Management.** A production manager can use OR techniques:

i. To find out the number and size of the items to be produced.

ii. In scheduling and sequencing the production runs by proper allocation of machines.

iii. In calculating the optimum product mix; and to select, locate, and design the sites for the production plants.

7.6 MODELS IN OPERATIONS RESEARCH

A model is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.

The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems.

Models can be classified according to following characteristics:

1. **Classification by Structure**

i. **Iconic Models-** Iconic models represent the systems as it is by scaling it up or down (i.e. by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model, of a real one. Other common examples of it are: photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct.

ii. **Analogue Models-** The models, in which one set of properties is used to represent another set of properties, are called analogue models. After the problem is solved, the solution is reinterpreted in terms of the original system.

For examples, graphs are very simple analogues because distance is used to represent the properties such as: time, number, per cent, age, weight, and many other properties.

iii. **Symbolic (Mathematical) Models-** The symbolic or mathematical model is one which employs a set of mathematical symbols (i.e. letters, numbers, etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equation to describe the behavior of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

2. Classification by Purpose

Models can also be classified by purpose of its utility. The purpose of a model may be descriptive, predictive or prescriptive.

i. **Descriptive Models-** A descriptive model simply describes some aspects of a situation based on observations survey, questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

ii. **Predictive Models-** Such models can answer 'what if' type of questions, i.e. they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

iii. **Prescriptive Models-** Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types:

i. **Deterministic Models-** Such models assume conditions of complete certainty and perfect knowledge. For example, linear programming, transportation and assignment models are deterministic type of models.

ii. **Probabilistic (or Stochastic) Models-** These types of models usually handle such situation in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, and sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behavior

i. **Static Models-** These models do not consider the impact of changes that take place during the planning horizon, i.e. they are independent of time.

ii. **Dynamic Models-** In these models, time is considered as one of the important variables and admits the impact of changes generated by time.

5. Classification by Method of Solution

i. **Analytical Models-** These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For example, a general linear programming model, the specially structured transportation and assignment models are analytical models.

ii. **Simulation Models-** A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions. Simulation modeling has the advantage of being more flexible than mathematical modeling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically.

7.7 PRINCIPLES OF MODELING

Following principles should be taken care of while building Operations Research models:

1. Do not build up a complicated model when a simple one would suffice.
2. Beware of modeling the problems to fit a technique.
3. Deductions must be made carefully.
4. Models should be validated prior to implementation.
5. A model should neither be pressed to do; nor criticized for failing to do that for which it was never intended
6. Beware of overselling the model in cases where assumption made for the construction of the model can be challenged.
7. The solution of a model cannot be more accurate than the accuracy of the information that goes into the construction.
8. Models are only aids in decision-marking.
9. Models should not be complicated. It should be as simple as possible.
10. Models should be as accurate as possible.

There are six important phases in OR study, but it is not necessary that in all the studies each and every phase is invariably present. These phases are arranged in following logical order.

Phase I- Observe the Problem Environment

Phase I in the process of OR study is observing the problem environment. The activities that constitute this phase are visits, conferences, observations, research and so on. With the help of such activities, the OR scientist gets sufficient information and support to proceed and is better prepared to formulate the problem.

Phase II- Analyse and Define the Problem

Phase II is analyzing and defining the problem. In this phase not only, the problem is defined, but also uses, objectives and limitations of the study are stressed in the light of the problem. The end result of this phase is a clear grasp of need for a solution and understanding its nature.

Phase III- Develop a Model

Phase III is to construct a model. A model is representation of some real or abstract situation. Operations research models are basically mathematical models representing systems, processes or environment in the form of equation, relationships or formulae. The activities in this phase include defining inter-relationships among variables, formulating equations, using known OR models or searching suitable alternate models. The proposed model may be field tested and modified in order to work under environmental constraints. The model may also be modified if the management is not satisfied with the answer that it gives.

Phase IV- Select an Appropriate Data Input

No model will work appropriately if data input is not appropriate. Hence, tapping the right kind of data is a vital phase in OR process. Important activities in this phase are analyzing internal-external data and facts, collecting opinions using computer data banks.

Phase V- Provide a Solution and Test Reasonableness

Phase V in OR process is to get a solution with the help of a model and data input. First, the solution is used to test the model and to find limitations, if any. If the solution is not reasonable or if the model is not behaving properly, updating and modification of the model is considered at this phase.

Phase VI- Implement the Solution

Finally, the tested results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts and those who are responsible for managing and operating the systems.

7.9 QUANTITATIVE TECHNIQUES OF OPERATIONS RESEARCH (OR MODELS)

A BRIEF ACCOUNT OF SOME OF THE IMPORTANT OR MODELS IS GIVEN BELOW:

1. **Distribution (Allocation) Models.** Distribution models are concerned with the allotment of resources so as to minimize cost or maximize profit subject to prescribed restrictions.
2. **Production/Inventory Models.** Inventory/ Production Models are concerned with the determination of the optimal (economic) order quantity and ordering (production) intervals considering the factors such as-demand per unit time, cost of placing orders, costs associated with good help up in the inventory and the cost due to shortage of goods.
3. **Queuing Models.** In queuing models an attempt is made to predict:
   i. How much average time will be spent by the customer in a queue?
   ii. What will be an average length of waiting line of queue?
   iii. What will be the traffic intensity of a queuing system?
4. **Markovian Models.** These models are applicable in such situations where the state of the system can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis. Brand- switching problems considered in marketing studies is an example of such models.
5. **Competitive Strategy Models (Games Theory).** These models are used to determine the behavior of decision-making under competition or conflict. Methods for solving such models have not been found suitable for industrial applications, mainly because they are referred to an idealistic world neglecting many essential features of reality.

6. **Network Models.** These models are applicable in large projects involving complexities and inter-dependencies of activities. Project Evaluation and Review Techniques (PERT) and Critical Path Method are used for planning, scheduling and controlling complex project which can be characterized as works.
7. **Job Sequencing Models.** These models involve the selection of such a sequence of performing a series of jobs to be done on machines that optimize the efficiency performance of the system.
8. **Replacement Models.** These models deal with the determination of optimum replacement policy in situations that arise when some items or machinery need replacement by a new one. Individual and group replacement policies can be used in the case of such equipments that fail completely and instantaneously.
9. **Simulation Models.** Simulation is a very powerful technique for solving much complex models which cannot be solved otherwise and thus it is being extensively applied to solve a variety of problems. In fact, such models are solved by simulation techniques where no other method is available for its solution.

7.10 GENERAL METHODS FOR SOLVING OPERATIONS RESEARCH MODELS

GENERALLY, THREE TYPES OF METHODS ARE USED FOR SOLVING OR MODELS.

1. **Analytic Method**. If the OR model is solved by using all the tools of classical mathematics such as differential calculus and finite differences available for the task, then such type of solutions are called analytic solutions. Solutions of various inventory models are obtained by adopting the so called analytic procedure.
2. **Iterative Method**. If classical methods fail because of complexity of the constraints or the number of variables, and then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until

either no further improvement is possible or the cost of further calculation cannot be justified.

3. **The Monte-Carlo Method.** The basis of Monte-Carlo technique is random sampling of variable's values from a distribution of that variable. The method involves taking sample observations, computing probability distributions for the variable using random numbers and constructing some functions to determine values of the decision variables.

Advantages:

i. These methods avoid unnecessary expenses and difficulties that arise during the trial and error experimentation.

ii. By this technique, we find the solution of much complicated mathematical expression which is not possible by any other method.

Disadvantages:

i. This technique does not give optimal answers to the problems. The good results are obtained only when the sample size is quite large.

ii. The computations are much complicated even in simple cases.

7.11 CHECK YOUR PROGRESS

1. Operations research is the application of ______________methods to arrive at the optimal Solutions to the problems.

   A. economical B. scientific C. a and b both

2. In operations research, the -----------------------------are prepared for situations.

   A. mathematical models B. physical models diagrammatic

   C. diagrammatic models

3. Operations management can be defined as the application of --to a problem within a system to yield the optimal solution.

   A. Suitable manpower

   B. mathematical techniques, models, and tools

C. Financial operations

4. Operations research is based upon collected information, knowledge and advanced study of various factors impacting a particular operation. This leads to more informed ---------------------------------.

A. Management processes B. Decision making

C. Procedures

5. OR can evaluate only the effects of -------------------------------------.

A. Personnel factors. B. Financial factors

C. Numeric and quantifiable factors.

6. By constructing models, the problems in libraries increase and cannot be solved.

A. True

B. False

7. Operations Research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations.

A. True

B. False

8. OR can be applied only to those aspects of libraries where mathematical models can be prepared.

A. True

B. False

9. The main limitation of operations research is that it often ignores the human element in the production process.

A. True

B. False

7.12 SUMMARY

OPERATIONS RESEARCH DEALS WITH THE APPLICATIONS OF SCIENTIFIC METHODS TO PROBLEM SOLVING. IT CAN BE USED IN ALMOST EVERY FIELD SUCH AS BUSINESS, AGRICULTURE, MANAGEMENT, ENGINEERING AND SCIENTIFIC APPLICATIONS. IT USES INTERDISCIPLINARY WHOLISTIC APPROACH TO FIND SOLUTION WHICH MAY NOT BE PERFECT BUT THE QUALITY OF SOLUTIONS CAN BE IMPROVED BY APPLYING OR TECHNIQUES. SOLVING A PROBLEM USING OR TECHNIQUES HAS SIX MAIN PHASES. MODELS USED IN OR CAN BE CLASSIFIED BASED ON STRUCTURE, PURPOSE, NATURE OF ENVIRONMENT, BEHAVIOUR, METHOD OF SOLUTION ETC. ANALYTIC, ITERATIVE AND MONTE-CARLO METHODS ARE GENERALLY USED TO SOLVE OR MODELS.

7.13 KEYWORDS

1. **Operation Research:-**Operations research is a discipline that deals with the application of advanced analytical methods to help make better decisions.
2. **Model:-**A model is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect. The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance.
3. **Principles of Modeling:-**There are different kind of principle user should remember while using modeling. Model are made easy to understand. They are not made complicated. They are made up to

7.14 SELF ASSESSMENT TEST

reality.

Q 1. WHAT IS OPERATIONS RESEARCH?

Q 2. What are the characteristics of OR?

Q 3. Discuss the limitations of operations research.

Q 4. Enumerate with brief description some of the techniques of OR.

Q 5. What is meant by a mathematical model of a real situation? Discuss the importance of models in the solution of operations research problems.

Q 6. State the different types of models used in operations research.

Q 7. Discuss the various classification schemes of models.

Q 8. Explain briefly the general methods for solving OR models.

Q 9. Describe the methodology of OR and enumerate the models used inproduction management.

Q 10. What are the steps involved in operations research?

Q 11. Describe briefly the different phases of operations research.

Q 12. Discuss the importance of operations research in decision-makingprocess.

Q 13. Discuss scientific method in OR.

Q 14. Discuss the significance and scope of operations research in modernmanagement.

Q 15. ‘Operations research is a bunch of mathematical techniques’. Comment.

Q 16. What are the essential characteristics of OR? Explain the role ofcomputers in this field.

Q 17. Write a note on application of various quantitative techniques indifferent fields of business decision-making

7.15 ANS TO CHECK YOUR PROGRESS

1. B

2. A

3. B

4. B

5. C

6. B

7. A

8. A

9. A

7.16 REFERENCES/SUGGESTED READINGS

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
COURSE CODE: MCA-25	AUTHOR: RENU BANSAL
LESSON NO. 8	

LINEAR PROGRAMMING

STRUCTURE

8.1 LEARNINGOBJECTIVE

The main objective of this lesson is to acquaint with the students with the concept of linear programming. Formulation of real life situations as linear programming problems and graphical solution of problems are explained through examples. This lesson also introduces the concept of slack and surplus variables which will be used throughout the book.

8.2 INTRODUCTION

IN 1947, GEORGE DANTZIG AND HIS ASSOCIATES, WHILE WORKING IN THE U.S. DEPARTMENT OF AIR FORCE, OBSERVED THAT A LARGE NUMBER OF MILITARY PROGRAMMING AND PLANNING PROBLEMS COULD BE FORMULATED AS MAXIMIZING/ MINIMIZING A LINEAR FORM OF PROFIT/COST FUNCTION WHOSE VARIABLES WERE RESTRICTED TO VALUES SATISFYING A SYSTEM OF LINEAR CONSTRAINTS (A SET OF LINEAR EQUATION/ OR INEQUALITIES). A LINEAR FORM IS A MATHEMATICAL EXPRESSION OF THE TYPE $a_1x_1 + a_2x_{2+\dots+}a_nx_N$,WHERE $a_1, a_2, \dots\dots a_n$ ARE CONSTANTS, AND $x_1, x_{2,\dots\dots,}x_N$ ARE VARIABLES. THE TERM 'PROGRAMMING' REFERS TO THE PROCESS OF DETERMINING A PARTICULAR PROGRAMME OR PLAN OF ACTION. SO LINEAR PROGRAMMING (L.P.) IS ONE OF THE MOST IMPORTANT OPTIMIZATION (MAXIMIZATION/MINIMIZATION) TECHNIQUES DEVELOPED IN THE FIELD OF OPERATIONS RESEARCH (O.R.).

The general Linear Programming Problem (LPP) calls for optimizing (maximization/minimization) a linear function of variables called the 'Objective Function' subject to a set of linear equation and /or inequalities called the 'Constraints' or 'Restrictions'.

8.3 FORMULATION OF LINEAR PROGRAMMING PROBLEMS

THE PROCEDURE FOR MATHEMATICAL FORMULATION OF LPP CONSISTS OF THE FOLLOWING STEPS:

Step 1 Identify the decision variables of the problem.

Step 2 Formulate the objective function to be optimized (maximized or minimized) as a linear function of the decision variables.

Step 3 Formulate the constraints of the problem such as resource limitations, market conditions, interrelation between variables and others as linear equation or inequations in terms of the decision variables.

Step 4 Add the non-negativity constraints so that negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraint and the non-negative constraint together form a linear programming problem.

The formulation of the LP problems is explained with the help of these examples.

Example 1 Production Allocation Problem

A firm manufactures two types of products, A and B and sells them at a profit of Rs 2 on type A and Rs 3 on type B. Each product is processed on two machines G and H. Type A requires one minute if processing time on G and two minutes on H; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day.

Formulate the problem as a linear programming problem.

Formulation

Decision Variables. Let x_1 be the number of products of Type A to be produced and let x_2 be the number of units of type B to be produced.

After carefully understanding the problem, the given information can be systematically arranged in the form of the following table.

Table 8.1

Machine	Time of Production (minutes)		Available Time (minutes)
	Type A($x_1$units)	Type B ($x_2$units)	
G H	1 2	1 1	400 600
Profit per unit	Rs.2	Rs. 3	

Objective function. Since the profit on type A is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 unites of type A. Similarly, $3x_2$ will be profit on selling x_2 units of type B. Therefore, total profit z on selling x_1 units of A and x_2 units of B is given by $\mathbf{z} = 2x_1 + 3x_2$

Hence, the objective function is

Maximize $\mathbf{z} = 2x_1 + 3x_2$.

Constraints. Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$. But machine G is available for not more than 6 hours 40 minutes (400 minutes).

Therefore $x_1 + x_2 \leq 400$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + x_2$, but machine H is available for 10 hours on a working day.

Therefore, $2x_1 + x_2 \leq 600$.

Non-negativity restrictions. Since it is not possible to produce negative quantities we have $x_1 \geq 0, x_2 \geq 0$.

Hence the allocation problem of the firm can be finally put in the form:

Find x_1 and x_2 such that the profit P is maximum i. e.

$$\textbf{Maximize } P = 2x_1 + 3x_2$$

Subject to the constraints

$$x_1 + x_2 \leq 400,$$

$$2x_1 + x_2 \leq 600,$$

$$x_1 \geq 0, x_2 \geq 0$$

Example 2 (Problem Mix Model)

A Company manufactures two bottling machines X, and Y. X is designed for 5-ounce bottles and Y for 10-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data are available.

Machine	5-ounce bottles	10-ounce bottles
X	80/min	30/min
Y	40/ min	50/min

The machines can be run 8 hours per day, for 5 days a week. Profit on 5-ounce bottles is 20 paise, and on 10-ounce bottles is, 30 paise. Weekly production of the drink cannot exceed 500,000 ounce; and, the market can absorb 30,000 (5-ounce) bottles and 8000 (10-ounce) bottles per week. The company wishes to maximize its profit, subject to all the production and marketing constraints.

Step 1. To determine the number of 5-ounce bottles and 10-ounce bottles to be produced per week let $x_1$andx_2 represent the number of bottles to be produced per week.

Step 2. The objective is to maximize the profit, that is, maximize z = Rs. ($0.20x_1 + 0.30x_2$).

Step 3. Constraints can be formulated as follows:

Since, a 5-ounce bottle takes 1/80 minutes and, a 10-ounce bottle takes 1/30 minutes on machine X, and the machine can run for 8 hours per day and 5 days per week, the time constraint on machine X is.

$$(x_1/80) + (x_2/30\ ) \leq 8\text{x}60\text{x}5 \leq 2400 \text{ minutes}$$

Similarly, the time constraint on machine Y is,

$$(x_1/40) + (x_2/50\ ) \leq 2400 \text{ minutes}$$

As, the total weekly production cannot exceed 500,000 ounces

$$5x_1 + 10x_2 \leq 500{,}000 \text{ ounces}$$

The constraints on market demand yield,

$$x_1 \leq 30{,}000, \qquad x_2 \leq 8000\text{bottles}$$

The complete product mix model is,

Maximize z= Rs. $(0.20x_1 + 0.30x_2)$.

Subject to

$$(x_1/80) + (x_2/30) \leq 8x60x5 \leq 2400$$

$$(x_1/40) + (x_2/50) \leq 2400$$

$$5x_1 + 10x_2 \leq 500{,}000$$

$$x_1 \leq 30{,}000, x_2 \leq 8000$$

Example 3 (Production Allocation Model)

A company manufactures two types of products. A and B and, sells them at a profit of Rs. 4 on type A and Rs. 5 on type B. Each product is processed on two machines, X and Y. Type A requires 2 minutes of processing time on X and 3 minutes on Y. Type B requires 2 minutes on X, and, 2 minutes on Y. The machine, X is available for not more than 5 hours 30 minutes, while Y is available for 8 hours during any working day. Formulate the problem as a LP problem.

Step 1. Let x_1 be, the number of products of type A, and x_2 be, the number of products of type B.

Step2. To determine the objective function.

The profit on type A is Rs.4 per product. Therefore, $4x_1$ will be the profit on selling $x_1$units of type A. Similarly, $5x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A, and x_2 units of B, is given by.

$$\boldsymbol{z} = 4x_1 + 5x_{2.}$$

Step3. To determine the constraints, since, machine X takes 2 minutes on type A and 2 minutes on type B, and, it is not available for more than 5 hours 30 minutes (i.e. 330 minutes), the constraint obtained is

$$2x_1 + 2x_2 \leq 330.$$

Similarly, since the machine Y is available for 8 hours (480 minutes), the constraint obtained is,

$$3x_1 + 2x_2 \leq 480.$$

The non-negativity restrictions are,

$$x_1 \geq 0 \text{ and } x_2 \geq 0.$$

The complete LP model is,

$$\text{Maximize } \mathbf{z} = 4x_1 + 5x_2.$$

Subject to

$$2x_1 + 2x_2 \leq 330$$

$$3x_1 + 2x_2 \leq 480$$

$$x_1 \geq 0\,,\ x_2 \geq 0$$

Example 4. A company produces two types of hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Formulation. Let the company produce hats of type A and hats of type B each day. So the profit P after selling these two products is given by the linear function:

P$=8x_1 + 5x_2$**(objective function)**

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by $2\mathrm{t}x_1 + \mathrm{t}x_2 \leq 500\mathrm{t}$ where t is the labour time per unit of second type i.e.

$$2x_1 + x_2 \leq 500.$$

But there are limitations on the sale of hats, therefore further restrictions are:

$$x_1 \leq 150,\ x_2 \leq 250.$$

Also, since the company cannot produce negative quantities,

$x_1 \geq 0,$ and $x_2 \geq 0.$

Hence the problem can be finally put in the form:

Find x_1 and x_2 such that the profit
$\mathbf{P = 8x_1 + 5x_2}$ **is maximum subject to the constraints**:
$\mathbf{2x_1 + x_2 \leq 500, x_1 \leq 150, x_2 \leq 250, x_1 \geq 0, x_2 \geq 0}$

Example 5. The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles- of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicine can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.

Formulate this problem as a L.P.P.

Formulation. Suppose the manufacturer produces x_1 and x_2 thousand of bottles of medicines A and B, respectively, since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill x_1 thousand bottles of medicine A will be $3x_1$ hours. Similarly, the time required to prepare x_2 thousand bottles of medicine B will be x_2 hours. Therefore, total time required to prepare x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will be $3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours. $3x_1 + x_2 \leq 66$.

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \leq 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \leq 20$ and $x_2 \leq 40$.

Number of bottles being non-negative, $x_1 \geq 0, x_2 \geq 0$

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become

P=8 x 1000 x_1 + 7x1000 x_2 or P=8000x_1 +7000x_2

> **Thus, the linear programming problem is:**
>
> **Max. P=8000x_1+7000x_2, subject to the constraints:**
>
> $3x_1 + x_2 \le 66, x_1 + x_2 \le 45, x_1 \le 20, x_2 \le 40$

Example 6. A toy company manufactures two types of doll, a basic version- doll A and a deluxe version—doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are available only 600 per day. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem.

Formulation. Let $x_1$and x_2 be the number of dolls produced per day of type A and B, respectively. Let the doll A requires t hrs so that the doll B requires 2t hrs. So the total time to manufacture $x_1$and x_2 dolls should not exceed 2,000t hrs. Therefore $tx_1 + 2tx_2 \le 2000t$. Other constraints are simple. Then the linear programming problem becomes:

> **Maximize** P=$3x_1 + 5x_2$
>
> **Subject to the constraints:**
>
> $x_1 + 2x_2 \le 2000$ **(time constraint)**
>
> $x_1 + x_2 \le 1500$ **(plastic constraint)**

8.4 GRAPHICAL SOLUTION OF TWO VARIABLE LINEAR PROGRAMMING PROBLEMS

SIMPLE LINEAR PROGRAMMING PROBLEMS OF TWO DECISION VARIABLES CAN BE EASILY SOLVED BY GRAPHICAL METHOD. THE OUTLINES OF GRAPHICAL PROCEDURE ARE AS FOLLOWS:

Step 1. Consider each inequality-constraint as equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality- constraint corresponding to that line is '≤', then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '≥', sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4. Obtain the solution points (the corner points of the feasible region).

Step 5. Calculate the values of objective function at the solution points.

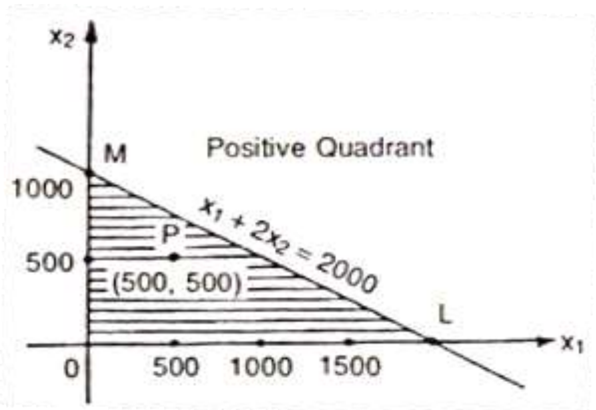
Step 6. For maximization problem, the optimum solution is the solution point which gives the maximum value of the objective function and for minimization problems the optimum solution is the solution point that gives the minimum value of the objective function.

Example 7. Find a geometrical interpretation and solution as well for the following LP problem: Maximize $z = 3x_1 + 5x_2$, subject to restrictions:

$x_1 + 2x_2 \leq 2000, x_1 + x_2 \leq 1500, x_2 \leq 600,$ and $x_1 \geq 0, x_2 \geq 0.$

Graphical Solution.

Step1. (To graph the inequality-constraints). Consider two mutually Perpendicular lines ox_1 and ox_2 as axes of coordinates. Obviously, any point (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions: $x_1 \geq 0, x_2 \geq 0$ to plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, $find\ x_1 = 2000$ from this equation.



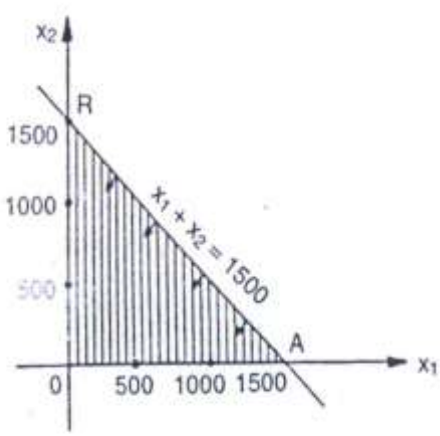
(Fig. 8.1)

Then mark a point L such that OL=2000. Similarly, again put $x_1 = 0$ to find $x_2 = 1000$ and mark another point M such that OM=1000.

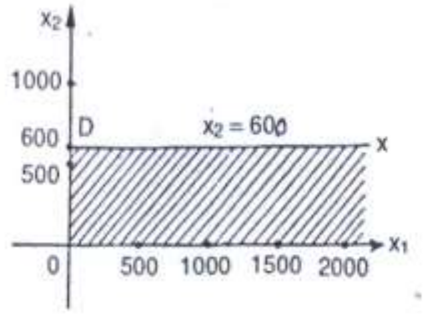
Now join the points L and M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in the above figure.

Clearly any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$.

Similar procedure is now adopted to plot the other two lines: $x_1 + x_2 = 1500$, and $x_2 = 600$ as shown in the figures.



(Fig. 8.2)



(Fig. 8.3)

Any point on or below the lines $x_1 + x_2 = 1500$, $x_2 = 600$ will also satisfy other two inequalities $x_1 + x_2 \leq 1500$, $x_2 \leq 600$ respectively.

Step 2. Find the feasible region or solution space by combining the figs. 8.1, 8.2 and 8.3 together. A common shaded area OABCD is obtained (see fig. 8.4) which is a set of points satisfying the inequality constraints:

$$x_1 + 2x_2 \leq 2000, x_1 + x_2 \leq 1500, x_2 \leq 600,$$

And non-negativity restrictions as $x_1 \geq 0, x_2 \geq 0$. Hence any point in the shaded area (including its boundary) is feasible solution to the given LPP.

Step 3. Find the co-ordinates of the corner points of feasible region O, A, B, C and D.

Step 4. Calculate the value of **z** for each corner point O, A, B, C and D. Maximum value of **z** is attained at the corner point B (1000,500), which is the point of intersection of lines

$x_1 + 2x_2 = 2000$ and $x_1 + x_2 = 1500$. Hence, the required solution is $x_1 = 1000$, $x_2 = 500$ and max value **z** = Rs. 5500.

Example 8. Consider the problem

$$\text{Max. } \mathbf{z} = x_1 + x_2,$$

Subject to the constraints:

$$x_1 + 2x_2 \leq 2000$$

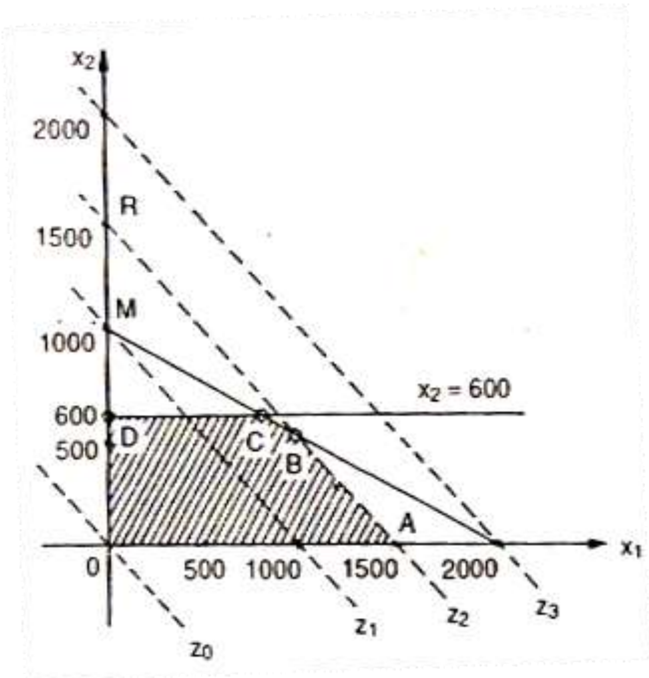
$$x_1 + x_2 \leq 1500$$

$$x_2 \leq 600$$

$$\text{and } x_1, x_2 \geq 0$$

Graphical Solution.

This problem is of the same type as discussed earlier except the objective function is slightly changed. The feasible region will be similar to that of the above problem.



(**Fig. 8.4**)

It is clear that the values of $x_1 and\ x_2$ which maximize value of **z** is always unique, but there will be an infinite number of feasible solutions which give unique value of **z**. Thus, two corners A and B as well as any point on the line AB give optimal solution of this problem.

It should be noted that if a linear programming problem has more than one optimum solution, there exists alternative optimum solutions. In the above example, one optimum solution is:

Maximum Profit **z** = Rs. 1500 at $x_1 = 1500\ and\ x_2 = 0$ (at corner point A). Alternative optimum solution to this problem is Maximum Profit **z** = Rs. 1500 at $x_1 = 1000\ and\ x_2 = 500$ (at corner point B).

Example 9. Old hens can be bought at Rs. 2 each and young ones at Rs.5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paisa. A hen (young or old) costs Rs. 1 per week to feed. I have only Rs. 80 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6 per week, assuming that I cannot house more than 20 hens.

Solution. Formulation, let x_1 be the number of old hens $and\ x_2$ the number of young hens to be bought. Since old hens lay 3 eggs per week and the young ones lay 5 eggs per week, the total number of eggs obtained per week will be $=3x_1 + 5x_2$.

Consequently the profit on each egg being 30 paisa, the total gain will be = Rs. 0.30 $(3x_1 + 5x_2)$.

Total expenditure for feeding $(x_1 + x_2)$ hens at the rate of Rs. 1 each will be = Rs. 1 $(x_1 + x_2)$.

Thus total profit **z** earned per week will be **z** = total gain- total expenditure

Or **z**=0.30$(3x_1 + 5x_2)$ - $(x_1 + x_2)$ or **z** = $0.50x_2$-$0.10x_1$ (objective function)

Since old hens can be bought at Rs. 2 each and young ones at Rs.5 each and there are only Rs. 80 available for purchasing hens, the constraints is :$2x_1 + 5x_2 \leq 80$.

Also since, is not possible to house more than 20 hens at a time, so $x_1 + x_2 \leq 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function **z** is to be maximized. Thus there is no need to add one more constraint, i.e. $0.5x_2$-$0.1x_1 \geq 6$.

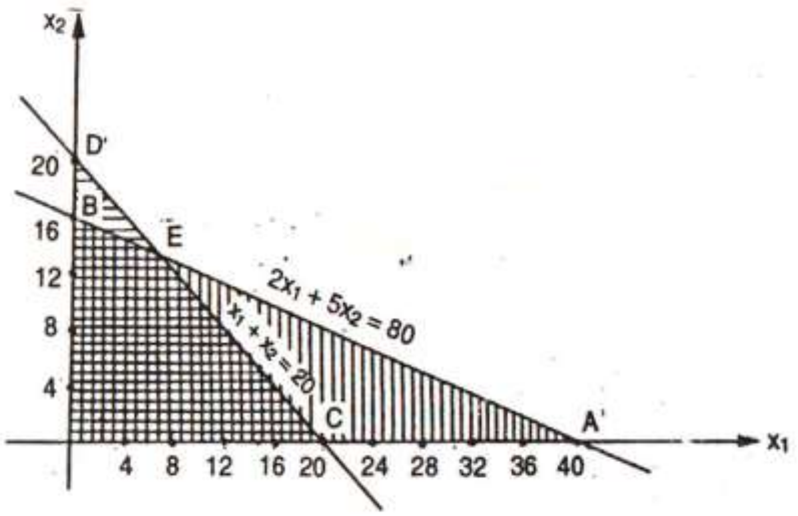
Again, it is not possible to purchase negative quantity of hens, therefore $x_1 \geq 0, x_2 \geq 0$.

Finally, the problem becomes:

Find x_1 and x_2 (real numbers) so as to maximize the profit function z = $0.50x_2 - 0.10x_1$

Subject to the constraints:

Graphical Solution. Plot the straight lines $2x_1 + 5x_2$=80 and $x_1 + x_2 = 20$ on the graph and shade the feasible region as shown in the figure.



(**Fig. 8.5**)

The feasible region is OBEC. The coordinates of the extreme points of the feasible region are:

O=(0,0), C=(20,0), B=(0,16), E=(20/3,40/3). The values of **z** at these vertices are:

$$z_A = 0$$

$$z_C = -2,$$

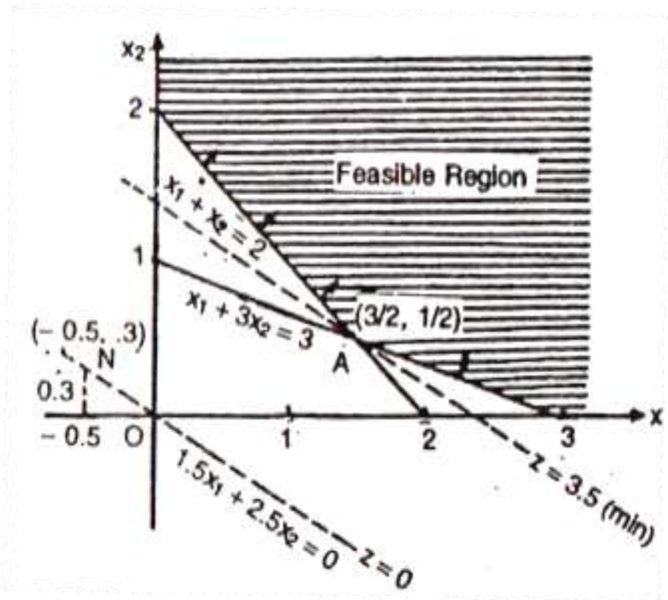
$$z_B = 8,$$

$$z_E = 6.$$

Since the maximum value of **z** is Rs. 8 which occurs at the point B= (0, 16), the solution to the given problem is $x_1 = 0, x_2 = 16$, max. **z** =Rs.8. Hence only 16 young hens I should buy in order to get the maximum profit of Rs. 8 (which is >6).

Example 10. (Minimization problem) Consider the problem: Min. z =1.5$x_1 + 2.5x_2$ subject to $x_1 + 3x_2 \geq 3, x_1 + x_2 \geq 2, x_1, x_2 \geq 0$.

Graphical Solution. The geometrical interpretation of the problem is given in following figure:



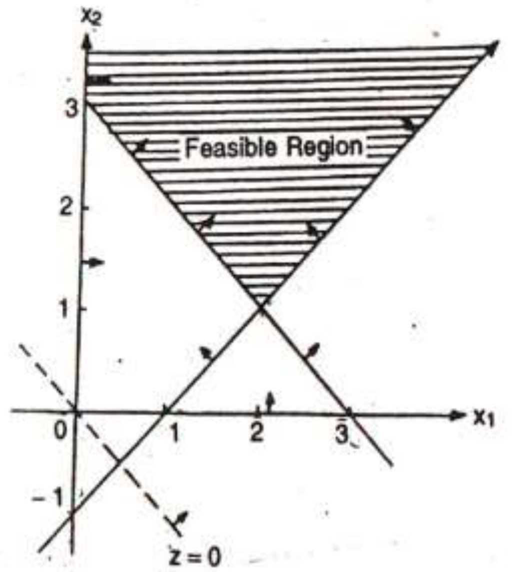
(**Fig. 8.6**)

The minimum value of z is z_A=3.5. This minimum is attained at the point of intersection A of the lines $x_1 + 3x_2 = 3, and\ x_1 + x_2 = 2$. This is the unique point to give the minimum value of z. Now, solving these two equations simultaneously, the optimum solution is :$x_1 = \frac{3}{2}, x_2 = \frac{1}{2}$ and min. z=3.5.

Example 11. (Problem having unbounded solution)

Max. z=3$x_1 + 2x_2$ subject to $x_1 - x_2 \leq 1, x_1 + x_2 \geq 3,$ and $x_1, x_2 \geq 0$.

Graphical Solution. The region of feasible solutions is represented by the shaded area in the given figure:



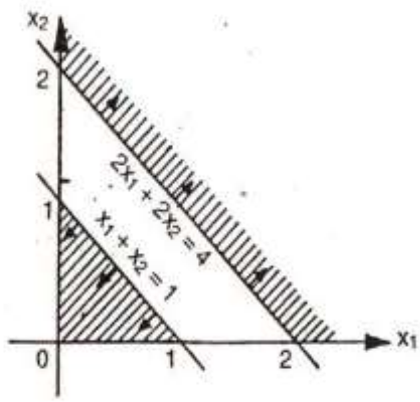
(**Fig. 8.7**)

It is clear from this figure that the feasible region is unbounded and the problem has no finite maximum value of **z**. Such problems are said to have unbounded solutions.

Example 12. (Problem with inconsistent system of constraints)

Maximize z=$3x_1 + 2x_2$ subject to $x_1 + x_2 \le 1, 2x_1 + 2x_2 \ge 4,$ and $x_1, x_2 \ge 0$.

Graphical Solution. The problem is represented graphically in figure.



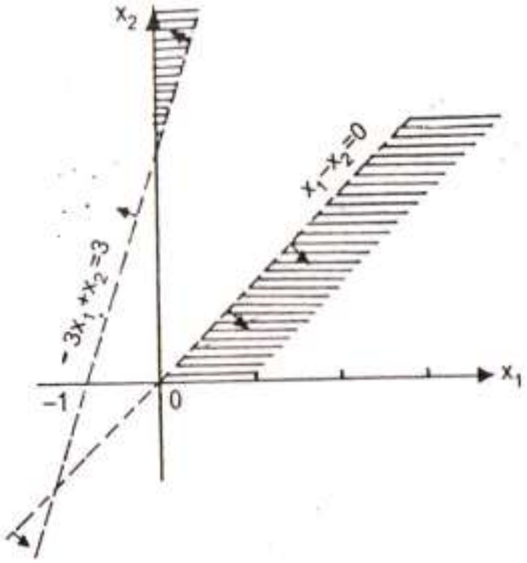
(**Fig. 8.8**)

The figure shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.

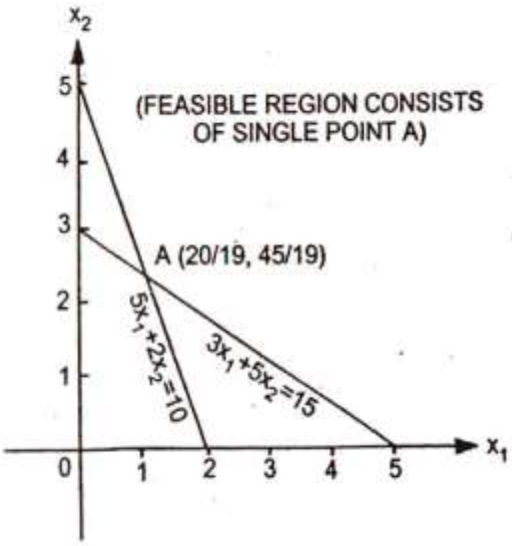
Example 13. (Constraints can be consistent and yet there may be no solution)

Max. z=$x_1 + x_2$ subject to $x_1 - x_2 \geq 0, -3x_1 + x_2 \geq 3,$ and $x_1, x_2 \geq 0.$

Graphical Solution. Figure for this example as given below shows that there is no region of feasible solutions in this case. Hence there is no feasible solution.



(**Fig. 8.9**)



(**Fig. 8.10**)

Example 14. (Problem in which constraints are equations rather than inequalities)

Max. z=$5x_1 + 3x_2$ subject to $3x_1 + 5x_2 = 15, 5x_1 + 2x_2 = 10,\ x_1 \geq 0, x_2 \geq 0.$

Graphical Solution. Figure above shows the graphical solution. Since there is only a single solution point A(20/19,45/19), there is nothing to be maximized.

8.5 GENERAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

THE GENERAL FORMULATION OF THE LINEAR PROGRAMMING PROBLEM CAN BE STATED AS FOLLOWS:

In order to find the values of n decision variables $x_1, x_{2,......,} x_n$ to maximize or minimize the objective function z = $c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_n x_n$

and also satisfy m-constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ (\leq = \text{ or } \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ (\leq = \text{ or } \geq) b_2$$

: : : : :

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\leq = \text{ or } \geq) b_i$$

: : : : :

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ (\leq = \text{ or } \geq) b_m,$$

where constraints may be in the form of any inequality (≤ or ≥) or even in the form of an equation (=), and finally satisfy the non-negativity restrictions.

$$x_1 \geq 0, x_2 \geq 0, \ldots . x_j \geq 0, \ldots . x_n \geq 0.$$

However, by convention, the values of right side parameters b_i (i = 1,2,3 …. , m) are restricted to non-negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constraint by -1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

8.6 SLACK AND SURPLUS VARIABLES

1. **Slack Variables.** If a constraint has ≤ sign, then in order to make it an equality, we have to add something positive to the left hand side. The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

For example, consider the constraints:

$$x_1 + x_2 \leq 2, 2x_1 + 4x_2 \leq 5, x_1, x_2 \geq 0$$

We add the slack variables $S_1 \geq 0, S_2 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$x_1 + x_2 + S_1 = 2$$

$$2x_1 + 4x_2 + S_2 = 5$$

$$x_1, x_2, S_1, S_2 \geq 0.$$

1. **Surplus Variables.** If a constraint has ≥ sign, then in order to make it an equality, we have to subtract something non-negative from its left hand side. Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example consider the constraints:

$$x_1 + x_2 \geq 2, 2x_1 + 4x_2 \geq 5, x_1, x_2 \geq 0.$$

We subtract the surplus variables $S_1 \geq 0, S_2 \geq 0$ from the left hand sides of above inequalities respectively to obtain.

$$x_1 + x_2 - S_1 \quad = 2$$

$$2\,x_1 + 4x_2 - S_2 = 5$$

$$x_1, x_2, S_1, S_2 \geq 0.$$

8.7 STANDARD FORM OF LINEAR PROGRAMMING PROBLEM

The standard form of the linear programming problem is used to develop the procedure for solving general linear programming problem. The characteristics of the standard form are explained in the following steps:

Step1. All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities (≥0). Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) the left side of each such constraint by non-negative variables. These new variables are called Slack Variables and are added if the constraints are (≤) or subtracted if the constraints are (≥).

For example, consider the constraints: $3\,x_1 - 4x_2 \geq 7,\ x_1 - 2x_2 \leq 3.$

These constraints can be changed to equations by introducing surplus and slack variables x_3 and x_4 respectively. Thus, we get.

$$3\,x_1 - 4x_2 - x_3 = 7,\ \ x_1 + 2x_2 + x_4 = 3, \text{and } x_3 \geq 0, x_4 \geq 0.$$

Step 2. The right side element of each constraint should be made non-negative (if not). The right side can always be made positive on multiplying both sides of the resulting equation by (-1) whenever it is necessary.

For example, consider the constraint as $3\,x_1 - 4x_2 \geq -4$

which can be written in the form of the equation $3\,x_1 - 4x_2 - x_3 = -4$ by introducing the surplus variable $x_3 \geq 0$.

Again, multiplying both sides by (-1), we get $-3\,x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step 3. Allvariables must have non-negative values.

A variable which is unrestricted in sign (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replace by $(x' - x")$, where x' and $x"$ are both non-negative, that is $x' \geq 0$ and $x" \geq 0$.

Step 4. The objective function should be of maximization form.

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $f(x)$, that is,

Min. $f(x) = -$ Max $\{-f(x)\}$

For example, the linear objective function

Min. z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

is equivalent to Max$(-$z), i.e. Max$z' = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$ with z= $-z'$.

Consequently, in any L.P. problem, the objective function can be put in the maximization form as given below:

Max. z $= c_1x_1 + c_2x_2 + \ldots + c_nx_n + ox_{n+1} + \cdots + ox_{n+m}$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n}x_n + x_{n+1}. \qquad = b_1$$

$$a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} \qquad = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n \qquad + x_{n+m} \quad = b_m$$

Where $x_1 \geq 0, x_2 \geq 0, \ldots\ldots x_n \geq 0, x_{n+1} \geq 0, \ldots., x_{n+m} \geq 0.$

Example 15 Express the following L.P. problem in standard form.

Min. z = $x_1 - 2x_2 + x_3$, subject to:

$2x_1 + 3x_2 + 4x_3 \geq -4,\ 3\,x_1 + 5x_2 + 2x_3 \geq 7,\ x_1 \geq 0,\ x_2 \geq 0$ and x_3 is unrestricted in sign.

Solution Proceeding according to the above rules, the standard LP form becomes:

Max (z')=$-x_1 + 2x_2 - (x_3{}' - x_3{}'')$, where $z' = -z$, subject to

$$-2x_1 - 3x_2 - 4(x_3{}' - x_3{}'') + x_4 \quad = 4$$

$$3x_1 + 5x_2 + 2(x_3' - x_3{}'') \qquad - x_5 = 7$$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3{}' \geq 0,\ x_3{}'' \geq 0,\ x_4 \geq 0,\ x_5 \geq 0.$

8.8 MATRIX FORM OF LINEAR PROGRAMMING PROBLEM

The linear programming problem in standard form can be expressed in matrix form as follows:

Maximize $z = CX^T$ **(objective function)**

Subject to $AX^T = b^T$, b ≥ 0 **(Constraint equation)**

X$\geq$ 0.**(non-negativity restriction)**

Where X= $(x_1, x_2, \ldots x_n, x_{n+1}, \ldots.,, x_{n+m})$,

C= $(c_1, c_2, \ldots c_n, 0,0, \ \ldots.,0)$ and b $= (b_1, b_2, \ldots b_m)$.

$$A = \begin{bmatrix} a_{11} a_{12} \ldots\ldots a_{1n} & 1 \; 0 \ldots 0 \\ a_{21} \quad a_{22}.\ldots \quad a_{2n} & 0 \; 1 \ldots 0 \\ \vdots \quad \vdots \quad \vdots & \vdots \; \vdots \ldots \vdots \\ a_{m1} \quad a_{m2}.\ldots \quad a_{mn} & 0 \; 0 \ldots 1 \end{bmatrix}$$

Example 16 Express the following LP problem in the matrix form.

Max. z =2 $x_1 - 3x_2 + 4x_3$, subject to

$x_1 + 2x_2 + x_3 \geq 5,\ x_1 + 2x_2 = 7,\ 5x_1 - 2x_2 + 3x_3 \leq 9,$ and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

Solution. This problem can be written in standard form as

Max. z =2 $x_1 + 3x_2 + 4x_3 + 0x_4 + 0x_5$

or

Max. z =(2, 3, 4, 0 , 0)$(x_1 x_2\ x_3\, x_4\ x_5)^T$

Subject to $x_1 + x_2 + x_3 - x_4 = 5,\ x_1 + 2x_2 = 7,\ 5x_1 - 2x_2 + 3x_3 + x_5 = 9$

Or

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Therefore, X= $(x_1 x_2\ x_3\, x_4\ x_5)^T$, C = (2 3 4 0 0), b =$(5\ 7\ 9)^T$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix}$$

8.9 SOME IMPORTANT DEFINITIONS

Some important definitions related to solution of linear programming problems are given below:

1. **Solution to a LPP** A set X=$\{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a solution to a LPP, if it satisfies the given set of constraints.

2. **Feasible solution** Any set x={$x_1, x_2, \ldots, x_{n+m}$} of variables is called a feasible solution of the LPP, if it satisfies the given set of constraints and non-negativity restrictions.

3. **Basis solution** A basis solution is a solution obtained by setting any n variables (among $m+n$ variables) equal to zero and solving for remaining m variables provided the determinant of the coefficients of these m variables is non-zero. Such m variables (any of them may be zero) are called basis variables, and the remaining variables which are set as zero are called non-basic variables.

   Thus, the number of basic solution must be almost $m+n$C$_m$.

4. **Basic feasible solution** A basic solution to a LPP is called as a basic feasible solution if it satisfies the non-negative restriction. There are two types of basic feasible solutions.

   (i) **Non-degenerate** All m basic variables are positive, and remaining n variables will be zero.

   (ii) **Degenerate** A basic feasible solution is degenerate, if one or more basic variables are zero.

5. **Optimum basic feasiblesolution** A basic feasible solution to a LPP is said to be its optimum solution if it optimizes (maximizes or minimizes) the objective function.

6. **Unbounded solution** If the value of the objective function z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

8.10 APPLICATIONS OF LINEAR PROGRAMMING

Some important applications of linear programming in real world are:

1. **Personnel Assignment Problem.** Linear programming can be applied for assignment of personnel (employees) for a particular task in some organization. The ratio in which skilled (trained) and unskilled (untrained) personnel are assigned for a particular task can be

determined by using linear programming which in turn reduces the overall cost and improves productivity.

2. **Transportation Problem.** The means of transportation of goods and the size and location of newly established warehouses can be determined by using linear programming.
3. **Agricultural Applications.** Linear programming can be applied in agricultural planning for allocating the limited resources such as acreage, labour, water, supply and working capital, etc. so as to maximize the net revenue.
4. **Military Applications.** These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defense against disaster to find the number of defence units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.
5. **Production Management.** Linear programming can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.
6. **Marketing Management.** Linear programming helps in analyzing the effectiveness of advertising campaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.
7. **Manpower Management.** Linear programming allows the personnel manager to analyses personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.
8. **Physical Distribution**. Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centers for physical distribution.

Besides above, linear programming involves the applications in the area of administration, education, inventory control, fleet utilization, awarding contract, and capital budgeting etc.

8.11 CHECK YOUR PROGRESS

1. WHAT IS THE OBJECTIVE FUNCTION IN LINEAR PROGRAMMING PROBLEMS?

   A. A constraint for available resource

   B. An objective for research and development of a company

   C. A linear function in an optimization problem

   D. A set of non-negativity conditions

2. Which statement characterizes standard form of a linear programming problem?

   A. Constraints are given by inequalities of any type

   B. Constraints are given by a set of linear equations

   C. Constraints are given only by inequalities of >= type

   D. Constraints are given only by inequalities of <= type

3. Feasible solution satisfies __________

   A. Only constraints B. only non-negative restriction

   C. [a] and [b] both D. [a],[b] and Optimum solution

4. In Degenerate solution value of objective function _____________.

   A. increases infinitely B. basic variables are nonzero

   C. decreases infinitely D. One or more basic variables are zero

5. Minimize Z = ______________

   A. –maximize(Z) B. -maximize(-Z)

   C. maximize(-Z) D. none of the above

6. In graphical method the restriction on number of constraint is __________.

   A. 2 B. not more than 3 C. 3 D. none of the above

7. In graphical representation the bounded region is known as _________ region.

A. Solution B. basic solution C. feasible solution D. optimal

8. Graphical optimal value for Z can be obtained from

A. Corner points of feasible region B. Both a and c

C. corner points of the solution region D. none of the above

9. In LPP the condition to be satisfied is

A. Constraints have to be linear B. Objective function has to be linear

C. none of the above D. both a and b

8.12 SUMMARY

The process of determining a particular programme or plan of action is called linear programming. Every linear programming problem has an objective function which is to be optimized (maximized or minimized) while satisfying the given set of constraints. The linear programming problems having two decision variables can be solved by graphical method. If the number of decision variables exceeds two, graphical method fails as the line of equation having more than two variables cannot be drawn on graph. Constraints are generally represented by inequalities whereas solving a given problem by graphical method requires the constraints in the form of equations. These inequalities can be converted into equations by using slack and surplus variables. Linear programming has a number of applications in our day-to-day life such as in business, agriculture, production, military etc.

8.13 KEYWORD

1. **Linear Programming; -**LINEAR PROGRAMMING (LP, also called LINEAR OPTIMIZATION) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by LINEAR relationships.

2. **Linear programming problem: -**A LINEAR PROGRAMMING PROBLEM consists of a LINEAR function to be maximized or minimized subject to certain constraints in the form of LINEAR equations or inequalities.

3. **Slack variable: -**In an optimization problem, a slack variable is a variable that added to an inequality constraint to transform it into an equality. Introducing a slack variable replaces an inequality constraint with an equality constraint and a non-negativity constraint on the slack variable.

4. **Surplus Variable: -**A **surplus variable** refers to the amount by which the values of the solution exceed the resources utilized. These **variables** are also known as negative slack **variables**. ... In order to obtain the equality constraint, the **surplus variable** is added to the greater than or equal to the type constraints.

8.14 SELF ASSESSMENT TEST

Q1. A manufacturer of Furniture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B.A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by manufacturer from chair and a table is Re. 1 and Rs. 5 respectively. What should be daily production of each of the two products?

Q2. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents, it is necessary to buy products (call them A and B) in addition. The contents of the various products, per unit, in nutrients are vitamins, proteins etc. is given in the following table:

Nutrients	Nutrients Contents in		Min. Amount of Nutrient
	A	B	
M_1	36	6	108

M_2	3	12	36
M_3	20	10	100

The last column of the above tables gives the minimum amount of nutrient constituents M_1, M_2, M_3 which must be given to the pigs. If the products A and B cost Rs. 20 and Rs. 4 per unit respectively, how much each of these two products should be bought so that the total cost is minimized?

Q 3. A company produces two types of leather belts, say type A and B. Belt A is of superior quality and belt B is of a lower quality. Profits on the two types of belt are 40 and 30 paise per belt respectively. Each belt of type A requires twice as much time as required by a belt of type B. If all belts were of type B, the company would produce 1,000 belts per day. But the supply of leather is sufficient only for 800 per day. Belt A requires a fancy buckle and 400 fancy buckles are available for this, per day. For bet of type B, only 700 buckles are available per day. How should the company manufacture the two types of belt in order to have maximum overall profit?

Q 4. A company sells two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit on products A and B respectively. The two products are produced in a common production process and sold in two different markets. The production process has a capacity of 30,000 man-hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8,000 and the maximum number of B is 12,000 units. Subject to these limitations, the products can be sold in any convex combinations. Formulate the above problem as a LPP and solve it by graphical method.

Q 5. A Manufacturer makes two products P_1 and P_2 using two machines M_1 and M_2. Product P1 requires 2 hours on machine M_1 and 6 hours on machine M_2. Product P_2 requires 5 hours on machine. M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production mix to optimize profit?

Q 6. Solve the following LP problems by graphical method:

a) Min. $z=5x_1 - 2x_2$; subject to $2x_1 + 3x_2 \geq 1, x_1, x_2 \geq 0$.

b) Max. $z=5x_1 + 3x_2$;subject to $3x_1 + 5x_2 \leq 15, 5x_1 + 2x_2 \leq 10;\ x_1, x_2 \geq 0$.

c) Max. $z=5x_1 + 7x_2$; s.t. $x_1 + x_2 \leq 4, 3x_1 + 8x_2 \leq 24,\ 10x_1 + 7x_2 \leq 35, x_1, x_2 \geq 0$.

d) Max. $z=2x_1 + 3x_2$;subject to $x_1 + x_2 \leq 1, 3x_1 + x_2 \leq 4;\ x_1, x_2 \geq 0$.

e) Min. $z=x_1 + 2x_2$;subject to $x_1 + 3x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1, x_2 \geq 0$.

8.15 ANSWER TO CHECK YOUR PROGRESS

1	C
2	A
3	C
4	D
5	B
6	D
7	C
8	A
9	D

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SUBJECT: DISCRETE MATHEMATICS AND OPTIMIZATION	
COURSE CODE: MCA-25	AUTHOR: MS. RENU BANSAL
LESSON NO. 9	
SIMPLEX METHOD & TWO-PHASE SIMPLEX METHOD	

STRUCTURE

9.1 LEARNING OBJECTIVE

The main objective of this lesson is to make the students learn about how to solve the linear programming problems having more than two decision variables. Two new iterative method called simplex method and Two- Phase Simplex Method is introduced in this lesson. The computational process of simplex method and Two- Phase Simplex Method is explained through suitable example and flowchart. The problem of degeneracy is also discussed.

9.2 INTRODUCTION

1. Introduction to Simplex Method

It is not possible to obtain the graphical solution to the LP problems having more than two decision variables. In such cases, a simple and most widely used simplex method is adopted which was developed by G. Dantzig in 1947. The Simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the **fundamental theorem of linear programming.**

The Simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of:-

(i) having a trail basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) Improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

2. Introduction to Two-Phase Simplex Method

Linear programming problems, in which constraints may also have '≥' and '=' signs after ensuring that all b_i are ≥0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the **artificial variable.** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in **Phase I** of the solution, and **Phase II** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called '**Two Phase Simplex Method'**.

Remarks:

1. The objective of Phase I is to search for a Basic Feasible Solution (B.F.S.) to the given problem. It ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.
2. The B.F.S. obtained at the end of Phase I provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.
3. In Phase II, care must be taken to ensure that an artificial variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artificial variable happens to leave the basis, its column must be deleted from the simplex table altogether.

9.3 DEFINITIONS AND NOTATIONS

Let x_B be a basic feasible solution to the LPP.

Maximize z = CX

Subject to AX = b

and, $X \geq 0$.

Then, the vector $C_b = (C_{b1}, C_{b2}, \dots, C_{bn})$ where C_{bi} are components of C associated with the basic variables, is called the cost vector associated with the basic feasible solution X_b.

Remarks:

1. If a LPP has a feasible solution, then it has a basic feasible solution.
2. There exists only finite number of basic feasible solutions to a LPP.

3. Let a LPP have a feasible solution. If we drop one of the basic variables and introduce another variable in the basic set, then the new solution obtained is also a basic feasible solution.

4. If the net evaluation $z_j - C_j = 0$ for at least one j for which $a_{ij} > 0, i = 1,2 \ldots, m$ then another basic feasible solution is obtained which gives an unchanged value of the objective function. If for at least one j, for which $a_{ij} \leq 0, i = 1,2 \ldots, m$ and $z_j - C_j$ is negative, then there does not exist any optimum solution.

Lastly, these notations can be summarized in the following Starting Simplex Table 9.1.

Table 9.1: Starting Simplex Table

$C_j C_1\ C_2$ … C_n 0 0 … 0

BASIC VARIABLES	C_b X_b	$X_1 = (a_1) X_2 = (a_2)$ … $X_n (= a_n)$ X_{n+1} X_{n+2} … X_n (B_1) (B_2) B_m	.. MIN RATIO
$X_{n+1} (=$ S $X_{n+2} (=$ S : $X_{n+m} (=$	$C_{b1} (= 0) X_{b1} (= b_1)$ $C_{b2} (= 0) X_{b2} (= b_2)$: : $C_{bm} (= 0) X_{bm} (= b_m)$	$X_{11} (= a_{11}) X_{12} (= a_{12})$… $X_{1n} (= a_{1n})$ 1 0 … 0 $X_{21} (= a_{21}) X_{22} (= a_{22})$… $X_{2n} (= a_{2n})$ 0 1 … 0 : : : : : : : : $X_{m1} (= a_{m1}) X_{m2} (= a_{m2})$… $X_{mn} (= a_{mn})$ 0 0 … 1	
	Z= $C_b X_b$	$\Delta 1 \Delta 2$ … Δn 0 0 …	$\Delta j = C_b X_j - C_j$

9.4 COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

Simplex method is an iterative procedure for solving the problems. Computational procedure of Simplex method involves the following steps:

Step 1. If the problem is one of minimization, convert it to a maximization problem by considering $-\mathrm{z}$ instead of z using the fact $\min \mathrm{z} = -\max.(-\mathrm{z})$ or $\min \mathrm{z} = -\max.(\mathrm{z}'), \mathrm{z}' = -\mathrm{z}$.

Step 2. We check up all $b_i{}'s$ for no negativity. If some of the are $b_i{}'s$ negative multiply the corresponding constraints through -1 by in order to ensure all $b_i \geq 0$.

Step 3. We change the inequalities to equations by adding slack and surplus variables, if necessary.

Step 4. We add artificial variable to those constraints with (≥) or (=) sign in order to get the identity basis matrix.

Step 5. We now construct the starting simplex table (Table 9.1). From this table, the initial basic feasible solution can be read off.

Form of Simplex Table (Summarized Table 9.1)

$c_j \;\; c_1 \;\; c_2 \;\; c_3 \ldots\ldots\ldots\ldots\; c_k \ldots\ldots\ldots\ldots\; c_{m+n}$

BASIC VARIABLES	$c_B \;\; x_B$	$x_1\, x_2\;\, x_3 \ldots\ldots\ldots\; x_k \ldots\ldots\ldots\; x_{m+n}$	MIN. RATIO RULE
---	----- -- ----	---- ---- ------- …….	….
	$z = c_B\, x_B$	$\Delta_1\Delta_2\Delta_3 \ldots\ldots\ldots\; \Delta_k \ldots\ldots\ldots\; \Delta_{m+n}$	← Δ_j

The computational aspect of the simplex procedure is explained by the following simple example.

Example 1. Consider the linear programming problem:

Maximize z =3 $x_1 + 2\,x_2$, subject to the constraints:

$$x_1 + x_2 \le 4,\; x_1 - x_2 \le 2, \text{and } x_1, x_2 \ge 0.$$

Solution:

Step 1. First, observe whether all the right side constants of the constraints are non-negative. If not it, can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the $b_1{}'$s(right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative. The coefficient of slack variables are always taken zero in the objective function. In this example, all inequality constraints being '≤', only slack variable s_1 and s_2 are needed. Therefore, given problem now becomes:

Maximize z =3 $x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix from:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 4. Construct the starting simplex table using the notations already explained in Section 3.2

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column x_B gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from Table 9.2 as: $s_1 = 4$, $s_2 = 2$, $x_1 = 0, x_2 = 0$, and the value of the objective function is zero.

Table 9.2: Starting Simplex Table for Example 1

c_j			3	2	0	0	
Basic Variables	c_B	x_B	x_1 (B_1)	x_2 (B_2)	x_3 (s_1)	x_4 (s_2)	MIN RATIO $x_B/\ x_K$ for $x_K > 0$
			MATRIX		BASIS		
s_1	0	4	1	1	1	0	To BE COMPUTD IN NEXT STEP
s_2	0	2	1	-1	0	1	
	$Z = c_B\, x_B$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 =$ 0	$\Delta_4 = 0$	$\Delta_J = z_J - c_j$ $= c_B\, x_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the 'net evaluation' Δ_J for each variable x_j(column vector x_j) by the formula

$$\Delta_J = z_J - c_j = c_B\, x_j - c_j$$

Thus, we get

$\Delta_1 = c_B\, x_1 - c_j$ $\Delta_2 = c_B\, x_2 - c_2$ $\Delta_3 = c_B\, x_3 - c_3$ $\Delta_4 = 0$

= (0,0)(1,1) –3 = (0,0)(1, –1) –2 =(0,0)(1,0) –0

= (0×1+0×1) – 3 = (0×1–0×1) – 2 =(0×1+0×0) – 0

=– 3 = – 2 =0

Optimality Test:

(i) If all Δ_j $(= z_J - c_j) \geq 0$, the solution under the test will be **optimal**. Alternative optimal solution will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded.**

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in **Step 6**.

Step 6. In order to improve this basic feasible solution, the vector entering the basic matrix and the vector to be removed from the basic matrix are determined by the following rules. Such vectors are usually named as 'incoming vector' and 'outgoing vector' respectively.

'Incoming vector'. The incoming vector x_k is always selected corresponding to the most negative value of Δ_j (say,Δ_k). Here Δ_k= min[Δ_1, Δ_2]= min [–3, –2]= –3=Δ_1. Therefore, k=1 and hence column vector x_1 must enter the basic matrix. The column x_1 is marked by an upward arrow (↑).

'Outgoing Vector'. The outgoing vector β_r is selected corresponding to the minimum ration of elements of x_B by the corresponding positive elements of predetermined incoming vector x_k . This rule is called the Minimum Ration Rule. In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min i\left[\frac{x_{Br}}{x_{ik}}, x_{ik} \geq 0\right]$$

For k=1, $$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}, \frac{2}{1}\right]$$

Or $$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this equation, we get r=2. So the vector β_2 i.e.,x_4 marked with downward arrow (↓) should be removed from the basic matrix. The **Starting Table 9.2** is now modified to **Table 9.3** given below.

	c_j		3 →	2	0	0	
Basic Variables	c_B	x_B	x_1	x_2	x_3 (s_1) (β_1)	x_4 (s_2) (β_2)	MIN. RATIO x_B/x_1
S_1	0	4	1	1	1	0	4/1

S_2	0 2	←□→ -------------1-----------------0----------- -------1----	--------------2/1 MIN. RATIO
	$Z = c_B\, x_B$=0	-3 -2 0 0 (MIN.Δ_J)	$\Delta_J = z_J - c_j$ $= c_B\, x_j - c_j$

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked ' ' □sition and zero at all other places of x_1 . IF the number in the marked ' ' □sition is other than unity, divide all elements of that row by 'key element'. (The element at the intersection of minimum ratio arrow ( ) and incoming vector arrow ( ) is called the **key element of pivot element**).

← ↑

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column x_1 . Thus, the process can be fortified by simple matrix transformation as follows:

The remaining coefficient matrix is:

X_B $X_1\, X_2\, X_3\, X_4$

R_1 4 $R_2$2	1 1	1 –1	1 0	0 1
R_3 $z = 0$	–3	– 2	0	0

Applying $R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 + 5R_1$

1 3	0 1	1 0	½ ½	−½ ½
$Z = 11$	0	0	5/2	½

Now construct the next improved simplex table as follows:

Table 9.4

c_j →3 2 0 0

Basic Variables	c_B x_B	x_1 x_2 (β_2) x_3 (s_1) (β_1) x_4 (s_2)	MIN. RATIO x_B / x_2 , $x_2 > 0$
S_1 S_2	0 2 3 2	0 □ 2 1 -1 □ 1 -1 0 1	2/2← key row 2/-1 (negative ration is not eounted
	$Z = c_B x_B$=6	0 − 5 ↑ 0 ↓ 3	← Δ_J

Key column

From this tables, the improved basic feasible solution is read as: $x_1 = 2, x_2 = 2, x_2 = 0, s_1 = 2, s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = c_B x_j - c_j$.

Step 8. Now repeat Steps 5 through 7 as and when needed until as optimum solution is obtained in table 3.5

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_{2.}$$

Therefore, k=2 and hence x_2 should be the entering vector (key column). By minimum ratio rule:

Minimum Ratio$\left(\frac{x_B}{x_2},\ x_2 > 0\right)$= Min $\left[\frac{2}{2}, -\right]$ (since negative ratio is not counted, so the second ratio is not considered and a ' – ' is put)

Since first ratio is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as:

	x_B	x_1	x_2	x_3	x_4
R_1	1	0	1	½	−½
R_2	2	1	−1	0	1
R_3	$Z = 6$	0	−5	0	3

Applying $R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 + 5R_1$

1	0	1	½	−½
3	1	0	½	½
$Z = 11$	0	0	5/2	½

Now construct the next improved simplex table as follows:

Table 9.5: Final Simplex Table

c_j 3 2 0 0

Basic Variables	c_B	x_B	x_1 (β_2)	x_2 (β_1)	s_1	s_2
→ x_2	2	1	0	1	½	−½
x_1	3	3	1	0	½	½
	$Z = c_B x_B$=11		0	0	5/2	½

The solution as read from this is : $x_1 = 3,\ x_2 = 1, s_1 = 0, s_2 = 0$, and max. $Z = 11$. Also, using the formula $\Delta_j = c_B x_j - c_j$ verify that all Δ_j are non − negative. Hence the optimum solution is

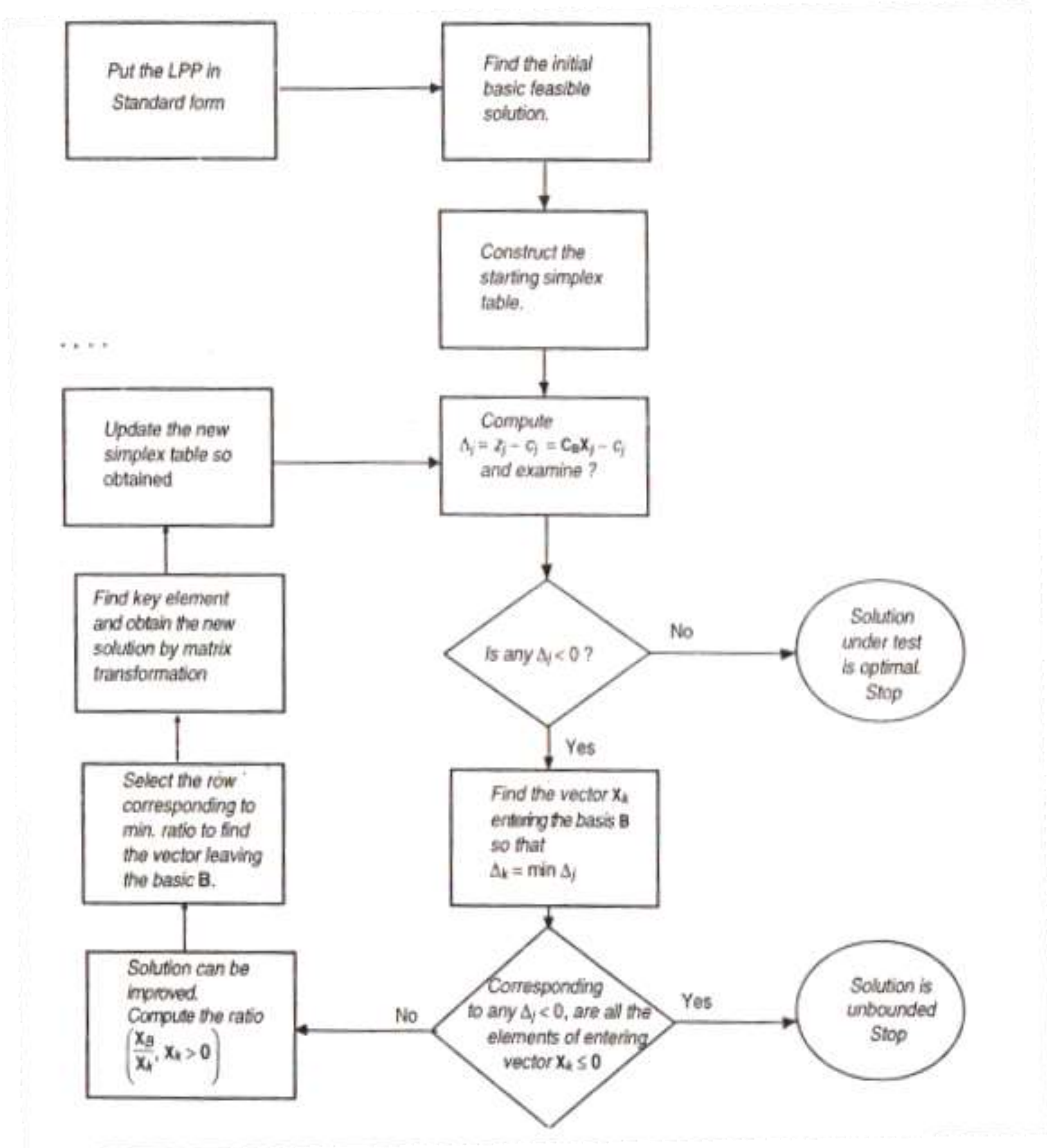
$$x_1 = 3,\ x_2 = 1, \max\ z = 11.$$

Tips for Quick Solution:

1. In the first nitration only, since Δ_j $'$s are the same as c_j $'$s, so there is no need of calculating them separately by using the formula $\Delta_j = c_B\, x_j - c_j$.
2. Mark min (Δ_j) by '↑' which at once indicates the column x_k needed for computing the minimum ratio (x_B / x_k).
3. 'Key element' is found at the place where the upward directed arrow '↑' of min Δ_j and the left directed arrow (←) of minimum ratio (x_B / x_k) intersect each other in the siplex table.
4. 'Key element' indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become 0.
5. Since Δ_j $'$s corresponding to unit column vectors are always zero, there is no need of calculating them.
6. While transforming the table by row operations, the value of z and corresponding are Δ_j' s also computed at the same time. Thus a lot of time and labour can be saved in adopting this technique.

Flowchart for Simple Simplex Method is given below:

9.5 FLOWCHART FOR SIMPLEX METHOD



9.6 DEFINITIONS USED FOR TWO-PHASE SIMPLEX METHOD

1. **Slack Variable.** As explained above, Slack Variable is a non-negative variable which is added to the left hand side of the constraint to convert it into equation while keeping the right hand side of the constraint positive.

For example, consider the constraint:

$$2x_1 + 3x_2 \leq 12,$$

We add the slack variables $s_1 \geq 0$ on the left hand side of above inequality to obtain the equation:

$$2x_1 + 3x_2 + s_1 \quad = 12$$

Where $s_1 \geq 0$.

2. **Surplus Variable.** For the constraints having sign ≥ while the right hand side of the constraint is positive, we have to subtract something non-negative from its left hand side. Thus the non- negative variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example consider the constraint:

$$x_1 + 2x_2 \geq 7.$$

We subtract the surplus variables $s_2 \geq 0$ from the left hand side of above inequality to obtain the equation.

$$x_1 + 2x_2 - s_2 \quad = 7$$

Where $s_2 \geq 0$.

3. **Artificial Variable.** Artificial Variable is used in the constraints having sign either ‘ = ‘ or ‘≥ ′ while keeping the right hand side of the constraint positive. For the constraints having sign ‘ = ‘ i.e. equations, only artificial variable is added to the left hand side;

whereas for the constraints having sign ‘ $\geq$ ‘, both the surplus and the artificial variables are used to make the constraint an equation. Artificial variable is used in Two- Phase Method (Artificial Variable Method) and Big- M Method (Charne’s Penalty Method) both of which are variants of Simple Simplex Method. Hence artificial variable is a non-negative variable which is added to the left hand side of the constraints having sign ‘ = ‘ or ‘$\geq$ ′.

If the right hand side of the constraint is negative, first convert it into positive by multiplying both sides by -1 and then introduce slack, surplus or artificial variables. For example, in the constraint $3x_1 - 2x_2 \leq -12$, the right hand side can be made positive by multiplying both sides by -1. Thus we get $-3x_1 + 2x_2 \geq 12$, now we can introduce surplus and artificial variables in it.

Some examples are given to make your understanding more clear.

Constraints	After introducing slack, surplus and artificial variables	Remarks
$3x_1 + 4x_2 + x_3 \leq 25$	$3x_1 + 4x_2 + x_3 + s_1 = 25$	(only slack variable is used)
$x_1 + 7x_2 \geq 7$	$x_1 + 7x_2 - s_1 + a_1 = 7$	(surplus and artificial variables are used)
$14x_1 + x_2 - 6x_3 = 7$	$14x_1 + x_2 - 6x_3 + a_1 = 7$	(only artificial variable is used)
$3x_1 - 2x_2 \leq -12$	$-3x_1 + 2x_2 - s_1 + a_1 = 12$	(surplus and artificial variables are used)

9.7 COMPUTATIONAL PROCEDURE OF TWO-PHASE SIMPLEX METHOD

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows:

Phase 1. In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost-1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise:

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^*=0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to Phase-II.

(iii) Max $z^*=0$ and no artificial vector appears in the optimum basis. In this case also proceed to Phase-II.

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of Phase-I, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of Phase-I are removed.

Flowchart for Two- Phase Simplex Mehtod is given below:

9.8 FLOWCHART FOR TWO-PHASE SIMPLEX METHOD

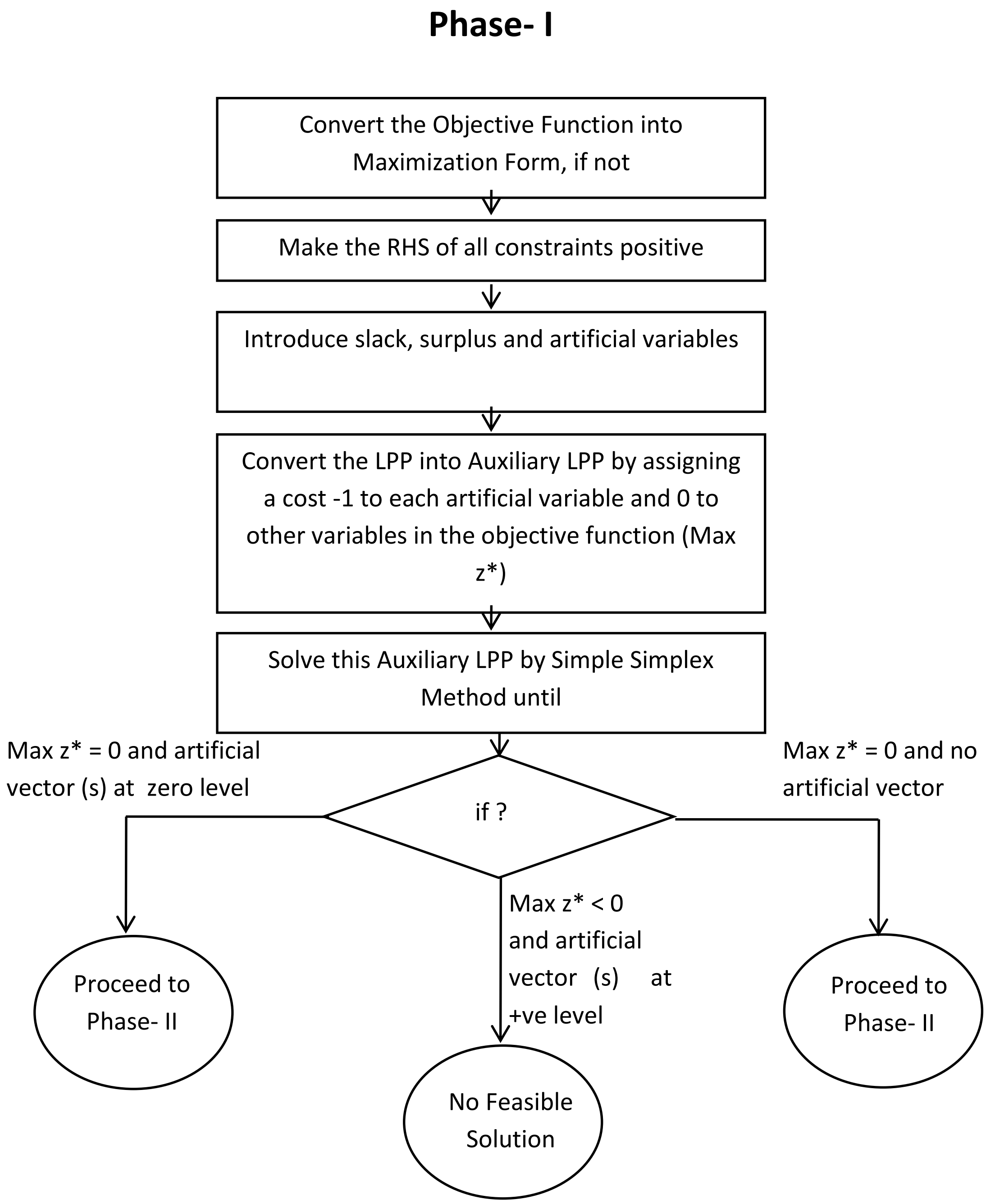


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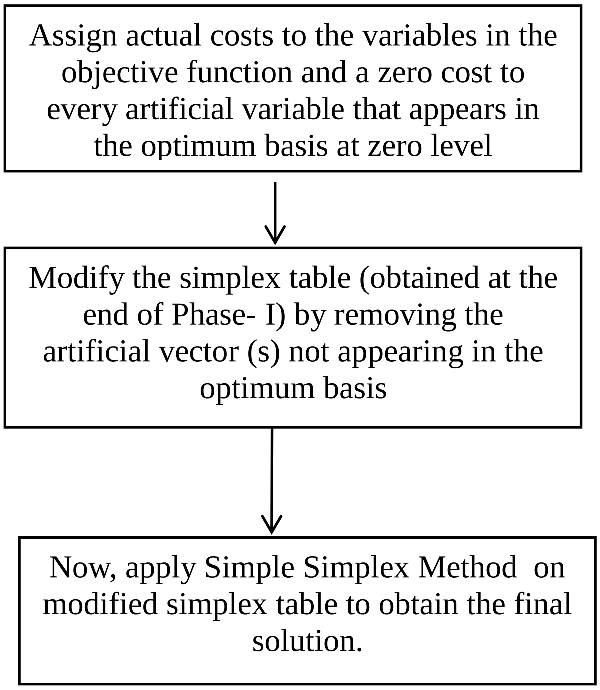
Fig. 9.8.1 Flowchart for Phase-I

PHASE -11



PHASE-II

Fig. 9.8.2 Flowchart for Phase-II

9.9 Solved Examples for Two-Phase Simplex Method

Example 1. Solve the problem: Min. $z = x_1 + x_2$ subject to

$$2\,x_1 + x_2 \geq 4, x_1 + 7x_2 \geq 7, and\ x_1, x_2 \geq 0.$$

Solution. First convert the problem of minimization to maximization by writing the objective function as :

Max. $(-z) = -x_1 - x_2$ or Max $z' = -x_1 - x_2$, where $z' = -z$.

Since all $b_i{}'$s (4 and 7) are positive , the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become:

$$2\,x_1 + x_2 - x_3 \qquad = 4$$

$$x_1 + 7x_2 - x_4 \qquad = 7$$

But the basis matrix B would not be an identity matrix due to negative coefficients of x_3 and x_4. Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraints equations can be written as

$$2\,x_1 + x_2 - x_3 + a_1 \qquad = 4$$

$$x_1 + 7x_2 - x_4 + a_2 = 7.$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (Table 9.8.1) where A_1 and A_2 denote the artificial column-vectors corresponding to a_1 and a_2 respectively.

Table 9.9.1

BASIC VARIABLES	x_B	x_1	x_2	x_3	x_4	A_1	A_2
a_1	4	2	1	−1	0	1	0
a_2	7	1	[7] ↑	0	−1	0	1 ↓
				×	×		

Now remove each artificial column vector A_1 and A_2 from the basis matrix. To remove vector A_2 first, select the vector either A_1 or A_2 , being careful to choose any one that will yield a non-negative revised solution. Take the vector x_2 to enter the basis matrix. It can be easily verified that if the vector A_2 is entered is place of x_1 , the resulting solution will not be feasible. Thus transformed table (Table 9.8.2) is obtained.

Table 9.9.2

BASIC VARIABLES	x_B	x_1	x_2	x_3	x_4	A_1	A_2
a_1	3	13/7	0	-1	1/7	1	$-1/7$
a_2	1	1/7	1	0	$-$ 1/7	0	1/7
					↑	↓	

This table gives the solution: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0$,$a_1 = 3, a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector A_2 again for re-entry into the basis matrix.

Similarly, remove A_1 from the basis matrix by introducing it in place of x_4 by the same method. Thus Table 9.8.3 is obtained.

Table 9.9.3

BASIC VARIABLES	x_B	x_1	x_2	x_3	x_4	A_1
x_4	21	13	0	-7	1	7
x_2	4	4	1	-1	0	1

This table give the solution: $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21, a_1 = 0$. Since the artificial variable a_1 become zero (non-basic), so drop the corresponding column A_1 from this table. Thus, the solution ($x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21$) is the basic feasible solution and now usual simplex routine can be started to obtained the required optimal solution.

Phase II. Now in order to test the starting solution for optimality, construct the starting simplex Table 9.8.4

Table 9.9.4

c_j → -1 -1 0 0

BASIC VARIABLES	c_B	x_B	x_1	x_2	x_3	x_4	MIN. RATIO (x_B/x_1)
← x_4	0	21	[13]	0	−7	1	21/13 ←
x_2	−1	4	2	1	−1	0	4/2
	$z' = C_B X_B = -4$		−1 ↑	0	1 ↓	0	← Δ_j

Compute $\Delta_1 = -1,\ \Delta_3 = 1.$

Key element 13 indicated that x_4 should be removed from the basis matrix. Thus, by usual transformation method Table 9.8.5 is formed.

Table 9.9.5

c_j → -1 -1 0 0

BASIC VARIABLES	c_B	x_B	x_1	x_2	x_3	x_4	MIN. RATIO COLUMN
→ x_1	−1	21/13	1	0	−7/13	1/13	
x_2	−1	10/13	0	1	1/13	−2/13	

	$z' = 31/13$	0 1/13	0	6/13	← $\Delta_j \geq 0$

Also, verify that

$$\Delta_j = C_B + X_3 - C_3 = (-1,-1)\left(-\frac{7}{13},\frac{1}{13}\right) = \frac{6}{13}$$

$$\Delta_j = C_B + X_4 - C_4 = (-1,-1)\left(\frac{1}{13},-2/13\right) = \frac{1}{13}.$$

Since all $\Delta_j \geq 0$, the required optimal solution is:

$$x_1 = \frac{21}{13}, x_2 = \frac{10}{13} \text{and min. z} = \frac{31}{13} (\text{because z} = \text{z}').$$

Simple Way for Two- Phase Simplex Method

Phase I: Table 9.9.6

BASIC VARIABLES	x_B	x_1 x_2 x_3 x_4	A_1 A_2
a_1 ← a_2	4 7	2 1 − 1 0 1 7 ↑ 0 − 1	1 0 0 1 ↓
← a_1 → x_2	3 1	13/7 0 − 1 1/7 1/7 1 0 ↑ − 1/7	1 − 1/7 0 1/7 ↓
← x_4 x_2	21 4	13 0 − 7 1	7 × 1

		2 1 − 1 0	×

Thus, initial basic feasible solution is : $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21$. Now start to improve this solution Phase II by usual simplex method.

Note.

1. Remove the artificial vector A_2 and insert it anywhere such that x_B remains feasible (≥0).
2. As soon as A_2 is removed from the basis by matrix transformation or otherwise, delete A_2 forever .
3. Similar process is adopted to remove other artificial vectors one by one from the basis.
4. Purpose of introducing artificial vectors is only to provide an initial basic feasible solution to start with simplex method in Phase II. So, as soon as the artificial variables become non-basic (i.e. zero), delete artificial vectors to enter Phase II.
5. Then, start Phase II, which is exactly the same as original simplex method.

Phase II: Table 9.9.7

c_j → -1− 1 0 0

BASIC VARIABLES	c_B x_B	x_1 x_2 x_3 x_4	MIN. RATIO x_B/ x_k
← x_4 x_2	0 2 1 −1 4	13 0 − 7 1 2 1 − 1 0	21/13 4/2
	$z' = -4$	− 1 * 0 1 0 ↑ ↓	← Δ_j
→ x_1 x_2	−1 21/13 −1 10/13	1 0 − 7/13 1/13 0 1 1/10	

		2/13	
	$z' = -31/13$	0 0 6/13 1/13	← $\Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = \frac{21}{13}, x_2 = \frac{10}{13}$, max. $z = 31/13$.

Example 2. Use two-phase simplex method to solve the problem: Minimize $z = x_1 - 2x_2 - 3x_3$, subject to the constraints:

$$-2x_1 + x_2 + 3x_3 = 2, 2x_1 + 3x_2 + 4x_3 = 1 \text{ and } x_1, x_2, x_3 \geq 0.$$

Solution. First convert the objective function into maximization form:

Max. $z' = -x_1 - 2x_2 + 3x_3$, where $z' = -z$.

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I. Auxiliary L.P. problem is : Max.$z'* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$ subject to above given constraints.

The following solution table is obtained for auxiliary problem.

Table 9.9.8

c_j 0 0 0 $-1-1$

BASIC VARIABLES	c_B x_B	x_1 x_2 x_3 A_1 A_2	MIN. RATIO x_B/x_k
a_1 ← a_2	-1 2 -1 1	-2 1 3 1 0 2 3 ↑ 4 0 1	2/3 1/4 ←

	$z' *= 3$	0 -4 $-7*$ 0 0	← Δ_j
a_1 → x_3	-1 5/4 0 1/4	$-$ 7/2 $-$ 5/4 0 1 $-$ 3/4 1/2 3/4 1 0 1/4	
	$z' *= -5/4$	7/4 5/4 0 0 3/4	← Δ_j ≥0

Since all Δ_j ≥0, an optimum basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time max. $z' *$ is negative and the artificial variable a_1 appears in the basic solution at a positve level. Hence the original problem does not possess any feasible solution. Here there is no need to enter Phase II.

Example 3. Use two-phase simplex method to solve the problem:

Minimize $z \ = \frac{15}{2}\, x_1 - \ 3x_2$, subject to the constraints:

$$3\, x_1 - x_2 \ - x_3 \ \geq 3, x_1 - x_2 \ + x_3 \ \geq 2,, \text{and } \ x_1, x_2\ , x_3 \ \geq 0.$$

Solution. Converty the objective function into the maximization form: maximize. $z' \ = -\frac{15}{2}\, x_1 + 3x_2$.

Introduciing the surplus variables $x_4 \geq 0$ and $x_5 \geq 0$, artificial variables $a_1 \geq 0, a_2 \geq 0$ the constraints of the given problem become

$$3\, x_1 - x_2 \ - x_3 - x_4 \quad + a_1 \quad = 3$$

$$x_1 - x_2 \ + x_3 \quad\quad - x_5 \quad + a_5 \ = 2$$

$$x_1, x_2\ , x_3\, , x_4\, , a_1, a_2 \ \geq 0.$$

Phase I. Assigning a cost -1 to artificial varibales a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary probelem becomes:

$Max. z' *= 0\, x_1 + 0x_2 \ + 0x_3 \ + 0x_4 \ + 0x_5 \ - 1a_1 \ - 1a_2$, Subject to the above given constraints.

Now apply simplex method in usual manner, (see table 9.9.9).

Phase I: Table 9.9.9

c_j 0 0 0 0 0 − 1 − 1

BASIC VARIABLES	c_B x_B	x_1 x_2 x_3 x_4 x_5	A_1 A	MIN. RATIO x_B/x_k
← a_1 a_2	−1 3 −1 2	3 −1 −1 −1 0 1 −1 1 0 −1	1 0 0 1	3/3 ← 2/1
	z′ *= −5	−4* 2 0 1 1 ↑	0 0 ↓	← Δ_j
→ x_1 ← a_2	0 1 −1 1	1 − 1/3 − 1/3 − 1/3 0 0 − 2/3 4/ 3 1/3 − 1	1/3 0 1/3 1	---- ¾ ←
	z′ *= −1	0 2/3 ↑ − 4/3 * 1/3 − 1	2/3 ↓ 0	← Δ_j
x_1 → x_3	0 5/4 0 3/4	1 − 1/2 0 − 1/4 − ¼ 0 − 1/2 1 1/4 − 3/4	¼ ¼ ¼ ¾	
	z′ *= 0	0 0 0 0 0	1 1	← Δ_j ≥0

Since all Δ_j ≥0 and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been atained.

Phase 2. In theis phase, now consider the actual costs associate with the original variables, the objective function thus becomes: $Max. z' = -15/2\, x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

Now apply simplex method in the usual manner.

Phase II: Table 9.9.10

c_j → $-15/2$ 3 0 0 0

BASIC VARIABLES	c_B	x_B	x_1	x_2	x_3	x_4	x_5	MIN. RATIO x_B/x_k
← x_1	$-15/2$	5/4	1	$-1/2$	0	$-1/4$	$-1/4$	
x_3	0	3/4	0	$-1/2$	1	1/4	$-3/4$	
.	$z' = -75/8$		0	3/4	0	15/8	15/8	← Δ_j

Since all $\Delta_j \geq 0$, an optimum basic feasibe solution has been attained.

Hence optimum solution is : $x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{3}{4}$, min z $= 75/8$.

Example 4. Max. $z = x_1 + 2x_2 + 3x_3 - x_4$, subject to the constraints:

$$x_1 + 2x_2 + 3x_3 = 15, 2x_1 + x_2 + 5x_3 = 20, x_1 + 2x_2 + x_3 + x_4 = 10, \text{and } x_1, x_2, x_3, x_4 \geq 0.$$

Solution. Introducing artificial variables a_1 and a_2 in the first and second constraint equations, respectively, and the original variable x_4 can be treated to work as an artificial variable for third constraint equation to obtain:

$$x_1 + 2x_2 + 3x_3 + a_1 = 15$$

$$2x_1 + x_2 + 5x_3 + a_2 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

Phase 1: Table 9.9.11

BASIC VARIABLES	x_B	x_1	x_2	x_3	x_4	A_1	A_2
a_1	15	1	2	3	0	1	0
a_2	20	2	1	5	0	0	
← x_4	10						

		1 1 2 1 1 0 0

By the same arguments as given in the previous examples of two-phase method insert x_4 in place of x_1. The transformed table (Table 9.9.12) is obtained by applying row transformation

$R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - 2R_3$.

Table 9.9.12

BASIC VARIABLES	x_B	x_1 x_2 x_3 x_4 A_1 A_2
a_1	5	0 0 2 −1 1 0
← a_2	0	0 −3 □ 3 −2 0 1
→ x_1	10	1 2 1 1 0 0

In spite of the fact that the artificial variable x_4 has served its puropse, the coloumn x_4 cannot be deleted from Table 4.12 because x_4 is the original variable also. Although the value of the artificial variable a_2 also becomes zero at this stage, the coloumn A_2 cannot be deleted unless it is inserted at one of the places x_2 or x_3 or x_4 . Now it is observed that A_2 can be inserted in place of x_3. Hence transformation Table 9.9.13 is obtained by applying the row transformations: $R_2 \rightarrow \frac{1}{3}R_2, R_1 \rightarrow R_1 - \frac{2}{3}R_2, R_3 \rightarrow R_3 - \frac{1}{3}R_2$.

Table 9.9.13

BASIC VARIABLES	x_B	x_1 x_2 x_3 x_4 A_1 A_2
← a_1	5	0 2 0 1/3 1 −2/3
→ x_3	0	
x_1	10	0 −1 1 −2/3 0 1/3

		1 3 0 5/3 0 $-4/3$

Now removing A_1 and inserting it in the suitable position of x_2 , the next transformed Table 9.8.14 is obtained by row transformations:

$R_1 \xrightarrow{} \frac{1}{2} R_1, R_2 \quad \overrightarrow{R_2} + \frac{1}{2} R_1, \; R_3 \quad \overrightarrow{R_3} - \frac{3}{2} R_1.$

Table 9.9.14

BASIC VARIABLES	x_B	x_1 x_2 x_3 x_4 A_1
x_2 x_3 x_1	5/2 5/2 5/2	0 1 0 1/6 1/2 0 0 1 $-1/2$ 1/2 1 0 0 7/6 $-3/2$

Delete column $A_1 (a_1 = 0)$. The starting basic feasible solutin is obtianed: $x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0.$

Further, proceed to test this solution for optimality in Phase II. For this, compute

$$\Delta_4 = c_B x_4 - c_4 = (2, 3, 1)\left(\frac{1}{6}, -\frac{1}{2}, \frac{7}{6}\right) - 0 = 0.$$

Phase II: Table 9.9.15

BASIC VARIABLES	c_B x_B	x_1 x_2 x_3 x_4	MINRATIO
x_2 x_3 x_1	2 5/2 3 5/2 1	0 1 0 1/6 0 0 1 $-1/2$ 1 0 0	

	5/2	7/6			
	$z = c_B\, x_B$ = 15	0 0*	0	0	← Δ_j

Since all Δ_j 's are zero, the solution: $x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0$, is optimal to give us z $*=$ 15. Further, being zero indicates that alternative optimal solutions are also possible.

Note. Here Δ_j corresponding to non basic vector x_4 also become zero. This indicates that alternative optimum solutions are possible. However, the other optimal solutions can be obtained as: $x_1 = 0, x_2 = \frac{15}{7}, x_3 = \frac{25}{7}, x_4 = 0, \text{max. z} = 15.$

Now, given the two alternative basic solutions;

(i) $x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0$ (ii) $x_1 = 0, x_2 = \frac{15}{7}, x_3 = \frac{25}{7}, x_4 = 0$

An infinite number of non-basic solutions can be obtained and by realizing them any weighted average of these two basic solutions is also an alternative optimum solution.

To verify this, third solution will be obtained as:

$x_1 = \frac{5/2+0}{2}$ = 5/4

$x_2 = \frac{5/2\ +15/7}{2}$ = 65/28

$x_3 = \frac{5/2+25/7}{2}$ = 85/28

$x_4 = \frac{0\ +\ 0}{2}$ = 0

and max. z = 15

9.10 PROBLEM OF DEGENERACY

At the stage of improving the solution, during simplex procedure, minimum ratio $x_B/x_k (x_k > 0)$ is determined in the last column of simplex table to find the key row (i.e.a row containing the key element). But sometimes this ratio may not be unique, i.e., the key element (here the variable to leave the basis) is not uniquely determined or at the very first iteration, the value of one or more basic variables in the x_B column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simple routine to cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon very rately occurs in practical problems.

Method to Resolve Degeneracy (Tie)

The following systematic procedure can be utilized to avoid cycling due to degeneracy in L.P. problems.

Step 1. First pick up the rows for which the min. non-negative ratio is same. To be definite, suppose such rows are first, third, etc., for example.

Step 2. Now rearrange the columns of the usual simplex table so that the column forming the original unit matrix come first in proper order.

Step 3. Then find the minimum of the ratio:

$$\left[\frac{elements\ of\ first\ column\ of\ unit\ matrix}{corresponding\ elementes\ of\ key\ column}\right],$$

Only if the rows for which min. ratio was not unique. That is, for the rows first, third, etc. as picked up in step1 (key column is the one for which Δ_j is minimum.)

(i) If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4. Now compute the minimum of the ratio:

$$\left[\frac{elements\ of\ second\ column\ of\ unit\ matrix}{corresponding\ elementes\ of\ key\ column}\right],$$

Only for the rows for which min. ratio was not unique in Step 3.

(i) If this min. ratio in unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5. Next compute the minimum of the ratio:

$$\left[\frac{elements\ of\ third\ column\ of\ unit\ matrix}{corresponding\ elementes\ of\ key\ column}\right],$$

Onlyfor the rows for which min. ratio was not unique in Step 4.

(i) If this min. ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this min. is still not unique, then go on repearting the above outlined procedure till the unique min. ratio is obtianed to resolve the degeneracy. After the resolution of this tie, simplex method is applied to obtain the optimum solution. Following example will make the procedure clear.

Example. Maximize $z = 3x_1 + 9x_2$, subject to the constraints: $x_1 + 4x_2 \leq 8, x_1 + 2x_2 \leq 4,$ and $x_1, x_2 \geq 0$.

Solution. Introducing the slack variables $s_1 \geq 0$and $s_2 \geq 0$, the problem becomes:

Max. $z = 3x_1 + 9x_2 + 0s_1 + 0s_2$ Subject to the constraints:

$$x_1 + 4x_2 + s_1 = 8$$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

Table 9.10.1

c_j 3 → 9 0 0

BASIC VARIABLES	c_B x_B	x_1 x_2 s_1 s_2	MIN. RATIO $(x_B/\ x_k)$

← s_1 s_2	0 8 0 4	1 4 1 0 1 2 0 1	$\left.\begin{matrix}\frac{8}{4}=2\\ \frac{4}{2}=2\end{matrix}\right\}$ Tie
	z = 0	−3 − 9 0 0	← Δ_j

Since min. ration 2 in the last column of above table is not unique, both the slack variables s_1 and s_2 may leave the basis. This is an indication for the existence of degeneracy in the given L.P. problem. So we apply the above outlined procedure to resolve degeneracy (tie).

First arrange the column x_1, x_2, s_1 and s_2 in such a way that the initial identity (basis) matrix appears first. Thus the initial simplex table becomes:

Table 9.10.2

c_j → 0 0 3 9

BASIC VARIABLES	c_B x_B	s_1 s_2 x_1 x_2	MIN. RATIO (s_1/ x_2)
s_1 ← s_2	0 8 0 4	1 0 1 4 0 1 ← 1 2	¼ 0/2 ←
	z = 0	0 ↓ 0 −3 ↑ − 9	← Δ_j ≥0

Now using the step 3 of the procedure for resolving degeneracy, we find

$$\min\left[\frac{elements\ of\ first\ column\ (s_1)}{corres.\ elements\ of\ key\ column\ (x_2)}\right] = \min\left[\frac{1}{4}, \frac{0}{2}\right] = 0$$

which occurs for the second row. Hence s_2 must leave the basis, and the key element is 2 as shown above.

First Iteration. By usual matrix transformation introduce x_2 and leave s_2.

Table 9.10.l3

c_j 0 → 0 3 9

BASIC VARIABLES	c_B x_B	s_1 s_2 x_1 x_2	MIN. RATIO
s_1 ← x_2	0 0 9 2	1 −2 −1 0 0 ½ ½ 1	
	z = 18	0 9/2 3/2 0	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum solution has been reached. Hence the optimum basic feasible solution is: $x_1 = 0, x_2 = 2, \max. z = 18.$

9.11 CHECK YOUR PROGRESS

1 Find Solution using Simplex Method.

Maximize $Z=3X_1+5X_2+4X_3$

Subject to the constraints

$2X_1+3X_2 \leq 8$

$2X_2+5X_3 \leq 10$

$3X_1+2X_2 +4X_3 \leq 15$

And $X_1, X_2, X_3 \geq 0$

2 Find Solution using Simplex Method.

Maximize $Z=4X_1+3X_2$

Subject to the constraints

$2X_1+X_2 \leq 1000$

$X_1+X_2 \leq 800$

$X_1 \leq 400$

$X_2 \leq 700$

And $X_1, X_2 \geq 0$

3 Find Solution using Simplex Method.

Maximize $Z=6X_1+4X_2$

Subject to the constraints

$2X_1+3X_2 \leq 30$

$3X_1+2X_2 \leq 24$

$X_1+X_2 \geq 3$

And $X_1, X_2 \geq 0$

4. Find Solution using Two-Phase Simplex Method.

Minimize $Z=X_1+X_2$

Subject to the constraints

$2X_1+4X_2 \geq 4$

$X_1+7X_2 \geq 7$

And $X_1, X_2 \geq 0$

5. Find Solution using Two-Phase Simplex Method.

Minimize $Z=3X_1+5X_2$

Subject to the constraints

$2X_1+8X_2 \geq 40$

$3X_1+4X_2 \geq 50$

And $X_1, X_2 \geq 0$

6. Find Solution using Two-Phase Simplex Method.

Minimize $Z=5X_1+8X_2$

Subject to the constraints

$3X_1+2X_2 \geq 3$

$X_1+4X_2 \geq 4$

$X_1+X_2 \leq 5$

And $X_1, X_2 \geq 0$

9.12 SUMMARY

Simplex Method:-The LPPs involving more than two decision variables cannot be solved using graphical method because solving the problem using graphical method involves drawing graph in the plane and a graph in two dimensional plane can be drawn for the equations/ inequalities having only upto two variables. The LPPs having more than two decision variables can be solved using Simplex Method. The Simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of:-

(i) having a trail basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

Simplex Method is also called Simple Simplex Method. Like Graphical Method, Simplex Method can also be used for solving LPPs having two decision variables but it can solve only the LPPs having all inequalities of type ($\leq$) and RHS of constraints all positive.

Two Phase Simplex Method:-The LPPs, in which constraints have the sign ' =' or '$\geq$' while keeping the right hand side constants positive, cannot be solved using Simple Simplex Method. To solve such problems, a variant of Simplex Method called the Two- Phase Method or the Artificial Method is used. Like Simple Simplex Method, Two- Phase Simplex Method can be used to solve LPPs having any number of decision variables. First of all objective function is converted into maximization form, if any. The right hand side of the constraints should be positive. If the right hand side of the constraint is not positive, it can be converted to positive value by multiplying both sides by -1. The sign '$\leq$' becomes '$\geq$' and the sign '$\geq$' becomes '$\leq$' . The sign ' = ' does not change in this process. Slack, surplus and artificial variables are used accordingly. Now Two- Phase Simplex method can be applied on this standard form. In Phase-I, an auxiliary LPP is constructed by assigning a cost -1 to artificial variables and zero cost to every other variables in the objective function. Now this auxiliary LPP is solved using Simple Simplex Method until one of the three possibilities arise- i) if Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution; ii) if Max $z^*=0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to Phase-II; and also iii) if Max $z^*=0$ and no artificial vector appears in the optimum basis. In this case also proceed to Phase-II.

In Phase- II, we assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of Phase-I, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of Phase-I are removed.

Simplex Method:-The SIMPLEX METHOD or SIMPLEX ALGORITHM is used for calculating the

9.13 KEYWORDS

optimal solution to the linear programming problem. In other words, the SIMPLEX ALGORITHM is an iterative procedure carried systematically to determine the optimal solution from the set of feasible solutions.A standard method of maximizing a linear function of several variables under several constraints on other linear functions.

Two-Phase Simplex Method: - the two-phase simplex method proceeds as follows:

1. Bring the constraints into equality form. For each constraint in which the slack variable and the right-hand side have opposite signs, or in which there is no slack variable, add a new artificial variable that has the same sign as the right-hand side.

2. Phase I: minimize the sum of the artificial variables, starting from the BFS where the absolute value of the artificial variable for each constraint, or of the slack variable in case there is no artificial variable, is equal to that of the right-hand side.

3. If some artificial variable has a positive value in the optimal solution, the original problem is infeasible; stop.

4. Phase II: solve the original problem, starting from the BFS found in phase I.

9.14 SELF ASSESSMENT TEST

Solve the following problems using Simplex Method.

Q 1. Max. $z = 2x_1 + x_2$, subject to

$4x_1 + 3x_2 \leq 12,$

$4x_1 + x_2 \leq 8,$

$4x_1 - x_2 \leq 8$

and $x_1, x_2 \geq 0.$

Q 2. Max. $z = 5x_1 + 3x_2$, subject to

$3x_1 + 5x_2 \leq 15,$

$5x_1 + 2x_2 \leq 10,$

and $x_1, x_2 \geq 0.$

Q 3. Max. $z = 7x_1 + 5x_2$, subject to

$-x_1 - 2x_2 \geq -6,$

$4x_1 + 3x_2 \leq 12,$

and $x_1, x_2 \geq 0.$

Q 4. Max. $z = 5x_1 + 7x_2$, subject to

$x_1 + x_2 \leq 4,$

$3x_1 - 8x_2 \leq 24,$

$10x_1 + 7x_2 \leq 35$

and $x_1, x_2 \geq 0.$

Q 5. Max. $z = 3x_1 + 2x_2$, subject to

$2x_1 + x_2 \leq 40,$

$x_1 + x_2 \leq 24,$

$2x_1 + 3x_2 \leq 60$

and $x_1, x_2 \geq 0.$

Q 6. Max. $z = 6x_1 + 4x_2$subject to

$2x_1 + 3x_2 \leq 30,$

$3x_1 + 2x_2 \leq 24,$

$x_1 + x_2 \geq 3$

and $x_1, x_2 \geq 0.$

Q 7. Max. $z = x_1 + 2x_2 + 3x_3 - x_4$, subject to the constraints:

$$x_1 + 2x_2 + 3x_3 = 15,$$

$$2x_1 + x_2 + 5x_3 = 20,$$

$$x_1 + 2x_2 + x_3 + x_4 = 10,$$

and $x_1, x_2, x_3, x_4 \geq 0$.

Q 8. Max. $z = 2x_1 + x_2$, subject to:

$$x_1 - x_2 \leq 10,$$

$$2x_1 - x_2 \leq 40$$

and $x_1 \geq 0, x_2 \geq 0$.

Q 9. Maximize $z = 107x_1 + x_2 + x_3$, subject to:

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7,$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5,$$

$$3x_1 - x_2 - x_3 \leq 0,$$

and $x_1, x_2, x_3 \geq 0$.

Q 10. Max. $z = 6x_1 - 2x_2$, subject to:

$$2x_1 - x_2 \leq 2,$$

$$x_1 \leq 4,$$

and $x_1, x_2 \geq 0$.

Q 11. Max. $z = 3x_1 + 2x_2$, subject to:

$$2x_1 + x_2 \leq 2,$$

$$3x_1 + 4x_2 \geq 12,$$

and $x_1, x_2 \geq 0$.

Q 12. Max. $z = 5x_1 - 2x_2 + 3x_3$, subject to

$$2x_1 + 2x_2 - x_3 \geq 2,$$

$$3x_1 - 4x_2 \leq 3,$$

$$x_2 + 3x_3 \leq 5$$

and $x_1, x_2, x_3 \geq 0.$

9.15 ANSWER TO CHECK YOUR PROGRESS

1. X_1=2.1707, X_2=1.2195, X_3=1.5122

   Max Z=18.6585

2. X_1=200, X_2=600

   Max Z=2600

3. X_1=8, X_2=0

   Max Z=48

4. X_1=1.6154, X_2=0.7692

   Min Z=2.3846

5. X_1=15, X_2=1.25

   Min Z=51.25

6. X_1=0, X_2=5

9.16 REFERENCES/SUGGESTED READINGS

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